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Quadratic Hermite–Padé polynomials associated with the exponential function

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Abstract

The asymptotic behavior of quadratic Hermite–Padé polynomials $p_n, q_n, r_n \in \mathscr{P}_n$ associated with the exponential function is studied for $n \to \infty$. These polynomials are defined by the relation

$$p_n(z) + q_n(z)e^z + r_n(z)e^{2z} = O(z^{3n+2}) \quad \text{as } z \to 0, \tag{$*$}$$

where $O(\cdot)$ denotes Landau's symbol. In the investigation analytic expressions are proved for the asymptotics of the polynomials, for the asymptotics of the remainder term in $(*)$, and also for the arcs on which the zeros of the polynomials and of the remainder term cluster if the independent variable z is rescaled in an appropriate way. The asymptotic expressions are defined with the help of an algebraic function of third degree and its associated Riemann surface. Among other possible applications, the results form the basis for the investigation of the convergence of quadratic Hermite–Padé approximants, which will be done in a follow-up paper.

MSC: 41A21; 30E10

Keywords: Quadratic Hermite–Padé polynomials of type I; Hermite–Padé polynomials of the exponential function; Hermite–Padé approximants

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1. Introduction

We are concerned with the asymptotic behavior of the Hermite–Padé polynomials (of type I) associated with the exponential function e^w. The investigation is based on a rescaling of the independent variable w. The rescaling method has already been used in an analogous way by Szegö [24] for the study of Taylor polynomials of the exponential function, and by Saff and Varga [20] for the study of Padé approximants again associated with the exponential function. We start with basic definitions and a short discussion of earlier research.

1.1. Hermite–Padé polynomials

In case of the exponential function the (diagonal) quadratic Hermite–Padé polynomials (of type I) $p_n, q_n, r_n \in \mathscr{P}_n$ are defined by the relation

$$e_n(w) := p_n(w) + q_n(w)e^w + r_n(w)e^{2w} = O(w^{3n+2}) \quad \text{as } w \to 0 \tag{1.1}$$

with e_n called the remainder term, and by $\mathscr{P}_n$ we have denoted the set of all polynomials of degree at most n. The aim of the paper is to improve the understanding of the asymptotic behavior of the Hermite–Padé polynomials p_n, q_n, r_n, and that of the remainder term e_n. Among other applications, the results are important for the study of the convergence of the quadratic Hermite–Padé approximants

$$\alpha_n(w) := \frac{1}{2r_n(w)}\left(-q_n(w) \pm \sqrt{q_n(w)^2 - 4p_n(w)r_n(w)}\right) \tag{1.2}$$

to the function e^w. Approximant (1.2) is an immediate consequence of (1.1) if one assumes the error term e_n to be negligible. The convergence of quadratic approximants will be investigated in a follow-up paper.

Quadratic Hermite–Padé approximants are a natural generalization of Padé approximants. They generalize Padé approximants in the same way as Padé approximants generalize Taylor polynomials, and they are also the first essential step towards the more general concept of algebraic Hermite–Padé approximants of given degree $m \geqslant 2$. In this more general framework, the quadratic Hermite–Padé polynomials p_n, q_n, r_n are a special case of the Hermite–Padé polynomials $p_{j,n_j} \in \mathscr{P}_{n_j}$, $j = 0, \ldots, m$, of type I and degree $m \geqslant 2$, which are defined by the relation

$$p_{0,n_0}(w) + p_{1,n_1}(w)e^w + \cdots + p_{m,n_m}(w)e^{mw} = O(w^{|n|+m}) \quad \text{as } w \to 0, \tag{1.3}$$

where $n = (n_0, \ldots, n_m) \in \mathbb{N}^{m+1}$ is a multi-index, $|n| := n_0 + \cdots + n_m$, and $m \geqslant 1$. The polynomials p_{j,n_j}, and a complementary set of polynomials q_{j,n_j}, $j = 0, \ldots, m$, of type II, have been introduced by Hermite, and are perhaps most famous for the role they have played in Hermite's proof of the transcendence of the number e in [8]. In Hermite's proof the polynomials q_{j,n_j} of type II have played the decisive role. A proof of the transcendence of e based on the polynomials p_{j,n_j} of type I was given later by Mahler [13].

Since these pioneering days the concept of Hermite–Padé approximants has kept a place in number theory (cf., for instance, [13–15,5,9]) and in approximation theory (cf. the survey about publications in approximation theory in [3], and a survey about asymptotic results in [1]). Detailed studies of quadratic approximants and especially of the polynomials p_n, q_n, r_n have been done in [4,6,7,21]. In [4], among other things, a 4-term recurrence relation together with very precise asymptotic estimates have been proved. While in [4], like in (1.1), only the diagonal case has been considered, the investigation has been extended to the non-diagonal situation in [6,7]. In [7] also very interesting connections with special functions have been established, and the paper contains results about the location of the zeros of all three polynomials p_n, q_n, and r_n. Actually, the plotting of the zeros of the middle polynomial q_n in [7] has triggered the interest of the author of the present paper. Especially the strange phenomenon that the zeros of q_n lie on a system of arcs that bifurcates on both ends (cf. Fig. 1) attracted his interest, and the phenomenon was not accessible to an easy and plausible explanation at the beginning of the investigations. A number of results from [6,7] have been extended to the general case (1.3) in [26]. In [27] these results have been used for an estimation of the irrationality of e.

1.2. Rescaling of the independent variable

Like in [4,6,7,26], we consider again the asymptotic behavior of the polynomials p_n, r_n, q_n, but now from a new and somewhat special point of view. The zeros of the polynomials p_n, r_n, and nearly all zeros of the polynomial q_n tend to infinity as

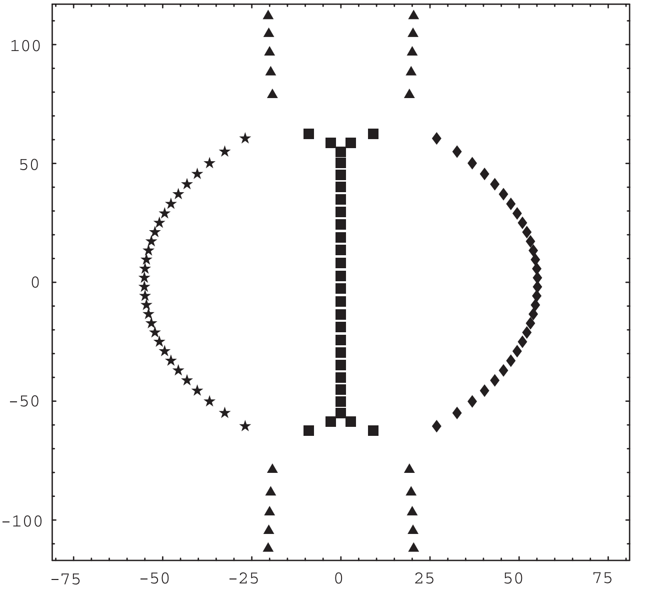

Fig. 1. The zeros of the polynomials p_{30} (stars), q_{30} (boxes), r_{30} (diamonds), and some of the zeros of the remainder term e_{30} (triangles). (Notice that the axes in both directions have different scales.)

$n \to \infty$. This divergence to infinity is not surprising, as the exponential function is entire, but for the investigation of the asymptotic behavior of the polynomials p_n, q_n, and r_n the divergence to infinity of most of the zeros is very unfortunate since many specific aspects of the asymptotic behavior are no longer accessible if almost all zeros cluster at infinity. It turns out that the asymptotic zero distributions and the asymptotics for the polynomials themselves become much more informative if the independent variable w is rescaled in such a way that the zeros of the transformed polynomials have finite cluster points as $n \to \infty$. We have already earlier mentioned that the concept of rescaling has been introduced by Szegö in [24] for the study of the asymptotic behavior of Taylor polynomials associated with the exponential function, and has also been used successfully by Saff and Varga in [20] for the study of zeros and poles of Padé approximants, again associated with the exponential function.

In the same spirit as in [20,24] we introduce as a new independent variable

$$z := \frac{w}{3n}, \quad n = 1, 2, \ldots, \tag{1.4}$$

for the study of the quadratic Hermite–Padé polynomials. Then the transformed polynomials P_n, Q_n, and R_n are defined by

$$P_n(z) := p_n(3nz), \quad Q_n(z) := q_n(3nz), \quad R_n(z) := r_n(3nz). \tag{1.5}$$

It is immediate that all asymptotic results that are proved in the variable z can easily be transferred to the situation of the original variable w. The new polynomials satisfy the relation

$$E_n(z) := P_n(z)(e^{-3z})^n + Q_n(z) + R_n(z)(e^{3z})^n = O(z^{3n+2}) \quad \text{as } z \to 0, \tag{1.6}$$

which, of course, is equivalent to relation (1.1). The remainder term E_n in (1.6) is connected with e_n by the relation $E_n(z) = e_n(3nz)e^{-3nz}$.

Note that in (1.6) not only the variable w has been substituted by $3nz$; the original relation (1.1) has also been multiplied by e^{-w} in order to make the inherent symmetry of the problem more evident. From (1.6) it immediately follows that $P_n(z) = R_n(-z)$ and $Q_n(z) = Q_n(-z)$.

The polynomials p_n, q_n, r_n, and P_n, Q_n, R_n are determined by (1.1) or (1.6) only up to a constant factor. This ambiguity is eliminated by assuming that the polynomial P_n is monic, i.e., that

$$P_n(z) = z^n + \cdots \quad \in \mathscr{P}_n. \tag{1.7}$$

Since the polynomials p_n, q_n, r_n form a perfect system (cf. [15]), the normalization (1.7) is always possible. (This possibility follows also from representation (1.11), below.)

Rescaling (1.4) could have been done with any multiple of n as denominator in (1.4) without changing the results in an essential way. The number $3n$ has been chosen since it is asymptotically equal to the number $3n + 3$ of free coefficients in relation (1.1) or (1.6).

The transformed approximant $A_n(z) := \alpha_n(3nz)$ is associated with the function $(e^z)^{3n} = \exp(3nz)$ in the same way as the approximant α_n, defined, in (1.2) is associated with the function e^z. We have

$$A_n(z) = \frac{1}{2R_n(z)}[-Q_n(z) \pm \sqrt{Q_n^2(z) - 4P_n(z)\,R_n(z)}]. \tag{1.8}$$

1.3. Methodological aspects

The zeros of the polynomials p_{30}, q_{30}, r_{30}, and some of the zeros of the remainder term e_{30} are shown in Fig. 1. These zeros, and of course in the same way also the zeros of the polynomials P_{30}, Q_{30}, R_{30}, and those of the remainder term E_{30}, form a very regular pattern, which certainly suggests that there should exist some analytic background for such a regular behavior. The main aim of the present paper is to prove analytic expressions for the asymptotic behavior of the polynomials P_n, Q_n, R_n, and the remainder term E_n. A core piece of these analytic expressions is an algebraic function ψ of third degree, and a harmonic function h defined on the Riemann surface $\mathcal{R}$ associated with ψ.

The asymptotics are derived by a saddle point method from the well-known (cf. [10]) integral representations:

$$R_n(z) = \frac{n!2^{n+1}e^{-3nz}}{2\pi i 3^n n^n} \oint_{C_1} \frac{e^{3nzv}\,dv}{v^{n+1}(v^2-1)^{n+1}}, \tag{1.9}$$

$$Q_n(z) = \frac{n!2^{n+1}}{2\pi i 3^n n^n} \oint_{C_0} \frac{e^{3nzv}\,dv}{v^{n+1}(v^2-1)^{n+1}}, \tag{1.10}$$

$$P_n(z) = \frac{n!2^{n+1}\,e^{3nz}}{2\pi i 3^n n^n} \oint_{C_{-1}} \frac{e^{3nzv}\,dv}{v^{n+1}(v^2-1)^{n+1}}, \tag{1.11}$$

$$E_n(z) = \frac{n!2^{n+1}}{2\pi i 3^n n^n} \oint_{C_\infty} \frac{e^{3nzv}\,dv}{v^{n+1}(v^2-1)^{n+1}}, \tag{1.12}$$

which have already been used by Hermite. Formulae (1.9)–(1.12) are slightly modified versions of the original formulae for p_n, q_n, r_n, and e_n given, for instance, in [10]. The modifications reflect substitution (1.4) and normalization (1.7). In (1.9)–(1.12) the integration paths C_1, C_0, C_{-1}, and C_∞ have to encircle the points $1, 0, -1, \infty$, respectively. The three first curves are positively and the fourth one is negatively oriented.

The asymptotics for the polynomials P_n, Q_n, R_n, and the remainder term E_n are certainly of interest in their own right, but they form also the basis for the investigation of the convergence behavior of the approximants A_n and α_n, which will be done in a follow-up paper. Of special interest is the phenomenon that the approximants A_n, and consequently also the algebraic approximants α_n, have branch points on the imaginary axis that do not correspond to singularities of the function to be approximated. Thus, they are an example for spurious branch points.

The phenomenon of spurious poles is well-known in the theory of Padé approximation (cf., for instance, [2] or [22]). It seems that the quadratic

Hermite–Padé approximants to the exponential function provide the first example of spurious branch points of algebraic Hermite–Padé approximants. The asymptotic behavior of quadratic Hermite–Padé polynomials of type II will be studied in a follow-up paper. First results are already contained in the paper [23].

After the present paper had been submitted for publication, another approach to the analysis of strong asymptotics of quadratic Hermite–Padé polynomials of type I has been undertaken in a paper by Kuijlaars et al., [11], which is based on a Riemann–Hilbert problem. An announcement of this result has already been published in [12].

The paper is organized as follows: Section 2 contains the statements and short discussions of all main results. In Section 3 results are proved which are related to the geometry of the arcs on which the zeros of the polynomials P_n, Q_n, R_n, and of the remainder term E_n cluster. The asymptotic relations for the polynomials P_n, Q_n, R_n, and the remainder term E_n are then proved in Section 4.

2. Main results

We start with theorems concerning the asymptotic distributions of the zeros and the nth root asymptotic behavior of the polynomials P_n, Q_n, R_n, and the remainder term E_n introduced in (1.6) and standardized by (1.7). The nth root asymptotic relations are preliminary versions of more precise results, which will be stated in Section 2.4 after several preparatory definitions have been made that are needed for the formulation of the strong asymptotic relations. These definitions involve a Riemann surface $\mathcal{R}$ and a harmonic function h defined on $\mathcal{R}$, both objects will be introduced in Section 2.2.

The weak versions of the asymptotic results have been placed in front, in order to give some orientation and also motivation for the more technical definitions, which follow in the Sections 2.2 and 2.3. The whole section is closed by two theorems containing tools for an efficient numerical calculation of the objects used in the asymptotic relations.

2.1. Weak versions of the asymptotic results

For a polynomial or an entire function F the (multi-)set of its zeros in $\mathbb{C}$ is denoted by $Z(F)$ and the zero counting measure v_F is defined as

$$v_F := \sum_{x \in Z(F)} \delta_x \tag{2.1}$$

with δ_x denoting Dirac's measure at $x \in \bar{\mathbb{C}}$. In case of an entire function F, the set $Z(F)$ can be infinite; multiplicities of zeros are represented by repetitions in $Z(F)$. By $\overset{*}{\to}$ we denote the weak convergence of measures in $\mathbb{C}$, i.e., we have $\mu_n \overset{*}{\to} \mu$ if, and only if, $\int f \, d\mu_n \to \int f \, d\mu$ for all continuous functions f with compact support in $\mathbb{C}$.

Theorem 2.1. *There exist three probability measures ν_j, $j=-1,0,1$, each with a compact support, and a positive measure ν_∞ with an unbounded support such that*

$$\frac{1}{n}\nu_{P_n} \overset{*}{\longrightarrow} \nu_{-1}, \quad \frac{1}{n}\nu_{Q_n} \overset{*}{\longrightarrow} \nu_0, \quad \frac{1}{n}\nu_{R_n} \overset{*}{\longrightarrow} \nu_1,$$
$$\frac{1}{n}\nu_{E_n} \overset{*}{\longrightarrow} \nu_\infty + 3\delta_0 \quad as\ n\to\infty. \tag{2.2}$$

It will be shown below in Lemma 2.6 that the supports of the measures ν_j, $j=-1,0,1,\infty$, consist either of analytic Jordan arcs or are the union of several analytic Jordan arcs. The measures ν_j are absolutely continuous with respect to these Jordan arcs, and for their density functions exist analytic expressions, which are given in Theorem 2.8, below. The arcs that form the supports of the measures ν_j, $j=-1,0,1,\infty$, are shown in Fig. 2, and a comparison of these arcs with Fig. 1, which is given in Fig. 3, shows that there is an impressive accordance.

The term $3\delta_0$ in the fourth limit of (2.2) stems from the zero of order at least $3n+2$, which the remainder term E_n has at the origin.

The asymptotics for the polynomials P_n, Q_n, R_n and the remainder term E_n given in the next theorem correspond in their precision with the results given in Theorem 2.1. Stronger versions of the asymptotic relations are contained in Theorem 2.9, below.

Theorem 2.2. *Locally uniformly we have*

$$\lim_{n\to\infty}\frac{1}{n}\log|P_n(z)| = \int \log|z-t|\,d\nu_{-1}(t) \quad for\ z\in\mathbb{C}\backslash\operatorname{supp}(\nu_{-1}),$$
$$\lim_{n\to\infty}\frac{1}{n}\log|Q_n(z)| = \log 2 + \int \log|z-t|\,d\nu_{0}(t) \quad for\ z\in\mathbb{C}\backslash\operatorname{supp}(\nu_{0}),$$

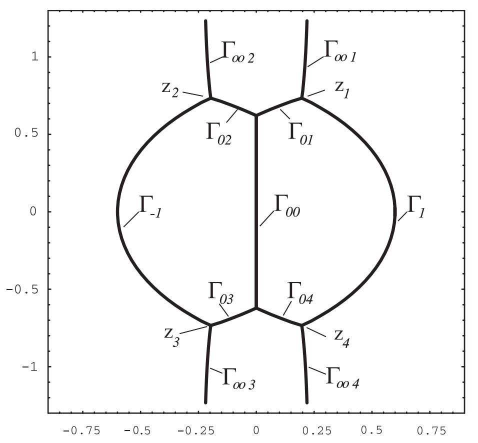

Fig. 2. The arcs Γ_{-1}, Γ_1, the set $K_0=\Gamma_{00}\cup\cdots\cup\Gamma_{04}$, and parts of the set $K_\infty=\Gamma_{\infty 1}\cup\cdots\cup\Gamma_{\infty 4}$.

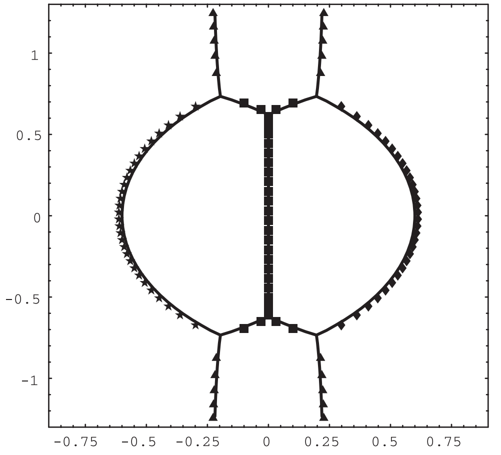

Fig. 3. An overlay of Figs. 2 and 1 after a shrinking of the scales of Fig. 1 in accordance with transformation (1.4).

$$\lim_{n\to\infty} \frac{1}{n} \log |R_n(z)| = \int \log |z-t|\, dv_1(t) \quad \textit{for } z\in\mathbb{C}\backslash \mathrm{supp}(v_1),$$

$$\lim_{n\to\infty} \frac{1}{n} \log |E_n(z)| = h(v_\infty; z) \quad \textit{for } z\in\mathbb{C}\backslash \mathrm{supp}(v_\infty), \tag{2.3}$$

with $h(v_\infty;\cdot)$ *defined by*

$$h(v_\infty; z) := 3\log|z| + \lim_{r\to\infty}\left[\int_{|t|\leqslant r} \log\frac{r|z-t|}{|r^2-\bar{t}z|}\, dv_\infty(t) + \right.$$
$$\left. -2\log r + \frac{1}{2\pi r}\int_{|t|=r} \max(-3\,\mathrm{Re}(t), \log(2), 3\,\mathrm{Re}(t))\frac{r^2-|z|^2}{|z-t|^2}|dt|\right]. \tag{2.4}$$

Remark. The measure v_∞ has an unbounded support and is of infinite mass. As a consequence its logarithmic potential is finite only after renormalization. In definition (2.4) of the function $h(v_\infty;\cdot)$ finiteness has been secured by a compensating term, which reflects the behavior of $h(v_\infty;\cdot)$ near infinity. Note that the last two terms in (2.4) are harmonic in $\{|z|<r\}$, which implies that all zeros of E_n are asymptotically represented by the measure $3\delta_0 + v_\infty$.

2.2. The Riemann surface $\mathcal{R}$ and the functions ψ and h

We start by defining a Riemann surface $\mathcal{R}$ together with an algebraic function ψ, which is then followed by the definition of a harmonic function h on $\mathcal{R}$. All three objects $\mathcal{R}$, ψ, and h are basic for the formulation of the strong asymptotic relations.

Definition 2.3. The Riemann surface $\mathscr{R}$ together with the bijective mapping $\psi : \mathscr{R} \to \bar{\mathbb{C}}$ and the canonical projection $\pi : \mathscr{R} \to \bar{\mathbb{C}}$ of the surface $\mathscr{R}$ onto $\bar{\mathbb{C}}$ is defined by the property that the function

$$z(v) := \frac{v^2 - 1/3}{v(v^2 - 1)}, \quad v \in \bar{\mathbb{C}}, \tag{2.5}$$

satisfies

$$z \circ \psi(\zeta) = \pi(\zeta) \quad \text{for all } \zeta \in \mathscr{R}. \tag{2.6}$$

The function ψ is algebraic of third degree. The surface $\mathscr{R}$ has three sheets and four simple branch points ζ_j, $j = 1, \dots, 4$, over the four base points

$$z_j = \pi(\zeta_j) := \sqrt[4]{1/3}\, e^{i\varphi_j} \quad \text{with } \varphi_j = \tfrac{5}{12}\pi, \tfrac{7}{12}\pi, \tfrac{17}{12}\pi, \tfrac{19}{12}\pi, \quad j = 1, \dots, 4. \tag{2.7}$$

Indeed, the derivative

$$z(v)' = -\frac{v^4 + 1/3}{v^2(v^2 - 1)^2} \tag{2.8}$$

has simple zeros at the four points $v_j = \sqrt[4]{-1/3}$, $j = 1, \dots, 4$, and it is easy to check that the four points (2.7) are defined by $z_j = \pi \circ \psi^{-1}(v_j) = z(v_j), j = 1, \dots, 4$, if the four roots v_j, $j = 1, \dots, 4$, are numbered in an appropriate way. The Riemann surface $\mathscr{R}$ is of genus 0.

Points on $\mathscr{R}$ will be denoted by ζ, while the associated base points $\pi(\zeta)$ will be denoted by $z \in \bar{\mathbb{C}}$. For shortness we call the points $\zeta_j, j = 1, \dots, 4$, and also their base points $z_j = \pi(\zeta_j)$, given in (2.7), branch points of $\mathscr{R}$. From (2.5) and (2.6) one easily deduces that

$$\psi \circ \pi^{-1}(\{0\}) = \{-\sqrt{1/3}, \infty, \sqrt{1/3}\} \quad \text{and} \quad \psi \circ \pi^{-1}(\{\infty\}) = \{-1, 0, 1\}. \tag{2.9}$$

We assume that the defining relation (1.6) of the Hermite–Padé polynomials P_n, Q_n, R_n is lifted to $\mathscr{R}$ so that a neighborhood of the origin in $\mathbb{C}$ corresponds to a neighborhood of the point ζ_0 on $\mathscr{R}$ with ζ_0 defined by

$$\zeta_0 := \psi^{-1}(\infty). \tag{2.10}$$

By definition we have $\pi(\zeta_0) = 0$, which is compatible with (2.10) because of (2.9).

In the next definitions and the analysis that follows, the function ψ will be of fundamental importance. For a given $\zeta \in \mathscr{R}$ the value $v = \psi(\zeta) \in \bar{\mathbb{C}}$ can be calculated very efficiently by solving the cubic equation

$$\pi(\zeta)\, v\, (v^2 - 1) - v^2 + \tfrac{1}{3} = 0. \tag{2.11}$$

Indeed, Eq. (2.11) has three solutions, and the value $v = \psi(\zeta)$ is one of them. However, the selection of the right solution is more difficult than solving Eq. (2.11). Any procedure for the selection defines a sheet structure on the surface $\mathscr{R}$, and on the other hand, if such a sheet structure has already been defined, then the selection is determined by the three sheets of $\mathscr{R}$. It turns out that for our purpose, i.e., for the asymptotic relations (2.33)–(2.36) in Theorem 2.9, below, a specific definition of sheets is needed. Their definition is implicitly already contained in relations (2.3) of Theorem 2.2, but an explicit definition of these sheets demands some preparations. It will be done with the help of a harmonic function h defined on $\mathcal{R}$, which will be introduced next after a short remark about Cardano's formulae.

Cardano's formulae give explicit solutions for Eq. (2.11). However, the sheet structure on the surface $\mathcal{R}$ defined by the solutions of Cardano's formulae is rather complicated, and we have not found any practical usefulness for them in connection with our asymptotic problems.

We now come to the definition of the functions h together with an analytic completed version h^* of h.

Definition 2.4. The functions $h^* : \mathcal{R} \to \bar{\mathbb{C}}$ and $h : \mathcal{R} \to \bar{\mathbb{R}}$ are defined as

$$h^*(\zeta) := u \circ \psi(\zeta) \quad \text{and} \quad h(\zeta) := \operatorname{Re} u \circ \psi(\zeta) \quad \text{for } \zeta \in \mathcal{R} \text{ with} \tag{2.12}$$

$$u(v) := \frac{2v^2}{v^2 - 1} + \log \frac{2}{3v(v^2 - 1)}. \tag{2.13}$$

It is immediate that the function h is harmonic in $\mathcal{R} \setminus (\{\zeta_0\} \cup \pi^{-1}(\{\infty\}))$ and subharmonic at ζ_0. The function h^* is an analytic completion of h, but h^* is not single-valued because of the logarithmic terms in (2.13).

In order to motivate the definition of the function h, we show in the next lemma that this function is already fully determined by requirements that are immediate consequences of the assumption that the three branches of the multivalued function $h \circ \pi^{-1}$ could be used for the representation of the right-hand sides of the asymptotic relations (2.3) in Theorem 2.2.

Lemma 2.5. *Let $\infty^{(-1)}, \infty^{(0)}, \infty^{(1)} \in \mathcal{R}$ denote the three points of $\pi^{-1}(\{\infty\})$ numbered in such a way that $\psi(\infty^{(j)}) = j$ for $j = -1, 0, 1$. If the function $\tilde{h}$ is harmonic in $\mathcal{R} \setminus \{\infty^{(-1)}, \infty^{(0)}, \infty^{(1)}, \zeta_0\}$ and satisfies*

$$\tilde{h}(\zeta) = -3 \operatorname{Re} \pi(\zeta) + \log|\pi(\zeta)| + O(1/\pi(\zeta)) \quad \text{as } \zeta \to \infty^{(-1)}, \tag{2.14}$$

$$\tilde{h}(\zeta) = \log|\pi(\zeta)| + O(1) \quad \text{as } \zeta \to \infty^{(0)}, \tag{2.15}$$

$$\tilde{h}(\zeta) = 3 \operatorname{Re}(\pi(\zeta)) + \log|\pi(\zeta)| + O(1) \quad \text{as } \zeta \to \infty^{(1)}, \tag{2.16}$$

$$\tilde{h}(\zeta) = 3 \log|\pi(\zeta)| + O(1) \quad \text{as } \zeta \to \zeta_0, \tag{2.17}$$

then $\tilde{h}$ is identical with the function h defined by (2.12) *and* (2.13).

Remark. If one wants to use branches of the multivalued function $h \circ \pi^{-1}$ for asymptotic representations of the four expressions $\frac{1}{n} \log|P_n(z)e^{-3nz}|$, $\frac{1}{n} \log|Q_n(z)|$, $\frac{1}{n} \log|R_n(z)e^{3nz}|$, and $\frac{1}{n} \log|E_n(z)|$ in the asymptotic relations (2.3), then it is clearly necessary that the three branch functions of $h \circ \pi^{-1}$ near infinity have to satisfy the three relations (2.14)–(2.16), and in addition one branch of the function $h \circ \pi^{-1}$ has to satisfy (2.17) at the origin. The same conclusions can be drawn directly from relation (1.6). Standardization (1.7) implies that in relation (2.14) the constant term is absent. Since we have assumed that relation (1.6) is lifted to $\mathcal{R}$ with the origin corresponding to ζ_0, relation (2.17) is a consequence of the zero of order at least $3n+2$ that the remainder term E_n has at the origin.

No proofs are given in the present section; all proofs are postponed to later sections. Thus, for instance, Lemma 2.5 will be proved in Section 3.

The introduction of the algebraic function ψ and the Riemann surface $\mathcal{R}$ in Definition 2.3 is somewhat unmotivated at the present stage. In any case, it can be accepted as an ansatz, which then will be justified in hindsight by the strong asymptotic results in Theorem 2.9, below.

That an algebraic function of third degree should be involved could also be deduced from the 4-term recurrence relations, which have been proved in [4] for the polynomials P_n, Q_n, R_n. Such a function corresponds also with a general conjecture in [16], where an algebraic function of degree $m+1$ has been introduced for the description of the asymptotic behavior of the Hermite–Padé polynomials p_{j,n_j} defined by relation (1.3).

2.3. Definition of the measures v_j

An analytic definition for the measures $v_j, j = -1, 0, 1, \infty$, which have appeared in Theorems 2.1 and 2.2, will be given together with an analytic description of the arcs that form the supports of these measures. The arcs are also the boundaries of domains $D_j, j = -1, 0, 1, \infty$, which will be introduced as domains of definition for branch functions h_j, h_j^*, and $\psi_j, j = -1, 0, 1, \infty$, of the multivalued functions $h \circ \pi^{-1}$, $h^* \circ \pi^{-1}$, and $\psi \circ \pi^{-1}$, respectively. These branch functions turn out to be building blocks for the strong versions of the asymptotic relations for the polynomials P_n, Q_n, R_n, and the remainder term E_n.

Associated with the function h from Definition 2.4 we consider the function

$$h_{\max}(z) := \max\{h(\zeta) \mid \zeta \in \mathcal{R}, \pi(\zeta) = z\}, \quad z \in \bar{\mathbb{C}}. \tag{2.18}$$

Note that this function is defined on $\bar{\mathbb{C}}$, while the function h had been defined on $\mathcal{R}$. Since h is harmonic in $\mathcal{R} \setminus (\{\zeta_0\} \cup \pi^{-1}(\{\infty\}))$ and subharmonic in a neighborhood of ζ_0, it follows from a standard result in potential theory (cf. [18, Chapter 2]) that the function $h_{\max}$, as the maximum of subharmonic functions, is also subharmonic. Further, we know from that result that $h_{\max}$ is harmonic at all $z \in \mathbb{C}$ except for those loci z, where at least two different branches of the multi-valued function $h \circ \pi^{-1}$ assume both the maximal value $h_{\max}(z)$. From the Poisson–Jensen formula of potential theory (cf. [18, Theorem 4.5.1]) we further know that because of its subharmonicity, the function $h_{\max}$ can be represented as the sum of a harmonic function and a Green potential. These details are formulated in the next lemma together with a description of the analytic arcs, on which the function $h_{\max}$ is not harmonic.

Lemma 2.6. *The function $h_{\max}$ is subharmonic in $\mathbb{C}$. There exists a system Γ of analytic Jordan arcs such that $h_{\max}$ is harmonic in $\mathbb{C}\setminus\Gamma$, but not harmonic in any neighborhood of a point $z\in\Gamma$. There exists a positive measure ν on Γ such that for any $R>0$ we have*

$$h_{\max}(z) = h_R(z) + \int_{|t|<R} \log\left|\frac{R(z-t)}{R^2-\bar{t}z}\right| d\nu(t) \quad \text{for } |z|\leqslant R \tag{2.19}$$

with h_R a function, which is harmonic in $\{|z|<R\}$ and satisfies

$$h_R(z) = h_{\max}(z) \quad \text{for } |z| = R. \tag{2.20}$$

The system of arcs Γ is symmetric with respect to the x- and the y-axis, and it can be broken down in subarcs such that

$$\Gamma = \Gamma_{-1}\cup K_0\cup\Gamma_1\cup K_\infty = K_{-1}\cup K_0\cup K_1\cup K_\infty, \tag{2.21}$$

where

(i) $K_{-1} = \Gamma_{-1}$ *is an analytic Jordan arc connecting the two branch points z_2 and z_3 in* $\{\mathrm{Re}(z)<0\}$,

(ii) $K_1 = \Gamma_1$ *is an analytic Jordan arc connecting the two branch points z_1 and z_4 in* $\{\mathrm{Re}(z)>0\}$,

(iii) K_0 *is the union of five analytic Jordan arcs $\Gamma_{00}, \Gamma_{01}, \dots, \Gamma_{04}$, where $\Gamma_{00} = [-iy_1, iy_1], y_1>0$, the two Jordan arcs Γ_{01} and Γ_{02} connect the two branch points z_1 and z_2 with the point iy_1, and the two Jordan arcs Γ_{03} and Γ_{04} connect the two branch points z_3 and z_4 with the point $-iy_1$, respectively. As a numerical value for y_1 we have*

$$y_1 \doteq 0.621391. \tag{2.22}$$

(iv) K_∞ *is the union of four analytic Jordan arcs $\Gamma_{\infty 1}, \dots, \Gamma_{\infty 4}$, which are disjoint in $\mathbb{C}$, and each arc $\Gamma_{\infty j}$ connects the branch point $z_j, j = 1, \dots, 4$, with infinity. The branch points $z_1, \dots, z_4$ have been defined in* (2.7).

Remarks. (1) All subarcs of Γ introduced in Lemma 2.6 are shown in Fig. 2. At each point of a subarc two branches of the function $h\circ\pi^{-1}$ assume the same value. If one crosses one of the subarcs of Γ, then two branches of $h\circ\pi^{-1}$ interchange their role as largest and second largest branch function of h in definition (2.18) of $h_{\max}$. Only at the three points $-iy_1, iy_1$, and ∞, all three branches of $h\circ\pi^{-1}$ have identical values. At all other points of Γ only two branches coincide, and the third one is smaller. At infinity the situation is special and more complicated since there all three branches of $h\circ\pi^{-1}$ have singularities.

(2) The function $h_{\max}$ can be calculated very efficiently for any given $z\in\mathbb{C}$ by solving Eq. (2.11) with $\pi(\zeta)$ replaced by z. If v_1, v_2, v_3 denote the three solutions of the equation, then

$$h_{\max}(z) = \max_{i=1,2,3} \mathrm{Re}\, u(v_j) \tag{2.23}$$

with u given by (2.13). Tools for an efficient numerical calculation of the subarcs of Γ are presented in Theorem 2.10, below.

(3) The expression $\log|(R^2-\bar{t}z)/(R(z-t))|$ in (2.19) is the Green function $g(z,t)$ in the disc $\{|z|<R\}$.

In the strong asymptotic relations of Theorem 2.9, below, we need branch functions of the multivalued function $h\circ\pi^{-1}$, which are defined in domains that are complementary to the sets K_j, $j=-1,0,1,\infty$, introduced in (2.21) of Lemma 2.6. Branch functions of the multivalued functions $\psi\circ\pi^{-1}$and $h_j^*\circ\pi^{-1}$ are defined in the same domains.

Definition 2.7. Let the domains $D_j, j=-1,0,1,\infty$, be defined as

$$D_j := \bar{\mathbb{C}}\backslash K_j, \quad j=-1,0,1,\infty, \tag{2.24}$$

with the sets K_j introduced in Lemma 2.6. Let further $\psi_j, j=-1,0,1,\infty$, be the branch functions of the multivalued function $\psi\circ\pi^{-1}$ that are defined in the domains D_j and satisfy

$$\psi_j(\infty)=j \quad \text{for } j=-1,0,1, \quad \text{and} \quad \psi_\infty(0)=\infty \quad \text{for } j=\infty. \tag{2.25}$$

In the same domains D_j, $j=-1,0,1,\infty$, branch functions π_j^{-1}, h_j, and $h_j^*, j=-1,0,1,\infty$, of the multi-valued function $\pi^{-1}, h\circ\pi^{-1}$, and $h^*\circ\pi^{-1}$, respectively, are defined by

$$\pi_j^{-1}=\psi^{-1}\circ\psi_j, \quad h_j=h\circ\pi_j^{-1}, \quad h_j^*=h^*\circ\pi_j^{-1} \quad \text{for } j=-1,0,1,\infty. \tag{2.26}$$

Remarks. (1) It is immediate that the functions h_j are harmonic in D_j for $j=-1,0,1$ and harmonic in $D_\infty\backslash\{0\}$ for $j=\infty$. Like-wise, the functions h_j^* and ψ_j are analytic in D_j for $j=-1,0,1$, and in $D_\infty\backslash\{0\}$ for $j=\infty$. Because of the logarithmic singularity at the origin, the function h_∞^* is not single-valued in $D_\infty\backslash\{0\}$.

(2) From the definition of the sets $K_j, j=-1,0,1,\infty$, in Lemma 2.6 it follows that

$$\bigcup_{j=-1,0,1,\infty} D_j = \bar{\mathbb{C}}\backslash\{z_1,\ldots,z_4\}. \tag{2.27}$$

Each point $z\in K_j\backslash\{z_1,\ldots,z_4\}, j=-1,0,1,\infty$, is covered by exactly three of the four domains $D_l, l=-1,0,1,\infty$, and each point $z\in\mathbb{C}\backslash\Gamma$, is covered by all four domains $D_j, j=-1,0,1,\infty$.

(3) Since the function $h\circ\pi^{-1}$ has three different branches at each $z\in\mathbb{C}\backslash\{z_1,\ldots,z_4\}$, at least two of the four values $h_j(z), j=-1,0,1,\infty$, have to be identical, and exactly two of them are identical at each point $z\in\mathbb{C}\backslash\{z_1,\ldots,z_4\}$, at which all three branches of $h\circ\pi^{-1}$ have different values. At each $z\in K_j\backslash\{z_1,\ldots,z_4\}, j=-1,0,1,\infty$, at least two of the three values $h_l(z), l\in\{-1,0,1,\infty\}\backslash\{j\}$, are identical. It follows from the proof of Lemma 3.3 in Section 3 that the two branch functions h_{l_1} and h_{l_2}, $l_1,l_2\in\{-1,0,1,\infty\}\backslash\{j\}$, with identical values on $K_j\backslash\{z_1,\ldots,z_4\}$ are the two branch functions with the largest values.

(4) The remark after Lemma 2.6 is also relevant with respect to the numerical calculations of the functions ψ_j, π_j^{-1}, h_j, and h_j^* introduced in Definition 2.7 since also here the calculation of $\psi \circ \pi^{-1}(z)$ is the key to all other calculations.

Next, we give an analytic definition for the four measures $\nu_j, j = -1, 0, 1, \infty$, that appear in the asymptotic relations (2.2) and (2.3) of the Theorems 2.1 and 2.2.

Theorem 2.8. (i) *The three restrictions*

$$\nu_j := \nu|_{K_j}, \quad j = -1, 0, 1, \tag{2.28}$$

of the positive measure ν *introduced in* (2.19) *of Lemma* 2.6 *are probability measures. The positive measure*

$$\nu_\infty := \nu|_{K_\infty} \tag{2.29}$$

has infinite mass.

The four measures $\nu_j,\ j = -1, 0, 1, \infty$, *defined by* (2.28) *and* (2.29) *are identical with the measures appearing in the Theorems* 2.1 *and* 2.2. *All four measures are absolutely continuous with respect to arc length, and for their density functions we have the representations*

$$\frac{d\nu_j(z)}{ds_z} = \frac{1}{\pi}\left[\frac{\partial}{\partial n_+}h_j(z) + \frac{\partial}{\partial n_-}h_j(z)\right], \qquad z \in K_j,\ j = -1, 0, 1, \infty, \tag{2.30}$$

with $\partial/\partial n_\pm$ *denoting the normal derivatives to both sides of the subarcs of the sets* $K_j,\ j = -1, 0, 1, \infty$, *and* ds_z *is the line element on these subarcs.*

(ii) *For the three functions* $h_j,\ j = -1, 0, 1$, *introduced in Definition* 2.7, *we have the representations*

$$h_j(z) = j3\,\mathrm{Re}(z) + \log\left(\frac{2}{j+1}\right) + \int \log|z-t|\, d\nu_j(t),$$
$$z \in \bar{\mathbb{C}},\ j = -1, 0, 1, \tag{2.31}$$

with the measures ν_{-1}, ν_0, ν_1 *defined by* (2.28), *and for the fourth function* h_∞ *we have the representation*

$$h_\infty(z) = 3\log\left|\frac{z}{R}\right| + h_R(z) + \int_{|t|<R} \log\left|\frac{R(z-t)}{R^2 - \bar{t}z}\right| d\nu_\infty(t) \quad \text{for } |z| \leqslant R \tag{2.32}$$

and any $R > 1$. *The function* h_R *is the same as that defined by* (2.20) *in Lemma* 2.6.

(iii) *We have* $h_\infty = h(\nu_\infty; \cdot)$ *with* $h(\nu_\infty; \cdot)$ *defined in* (2.4) *of Theorem* 2.2.

Remark. Since the measure ν_∞ is of infinite mass, representation (2.32) holds only for $R < \infty$. If $R \to \infty$, then the function h_R and also the Green potential in (2.32) tend to infinity, but both functions with opposite signs, and their sum remains finite for all $z \in \mathbb{C} \backslash \{0\}$.

With the definitions of the functions $h_j, h_j^*, \psi_j, j = -1, 0, 1, \infty$, and their domains of definition $D_j, j = -1, 0, 1, \infty$, in Definitions 2.7 we are prepared to formulate the strong asymptotic relations.

2.4. Strong asymptotic results

Strong versions of the asymptotic relations for the polynomials P_n, Q_n, R_n, and the remainder term E_n are presented in the next theorem. These relations are the main result of the present paper.

Theorem 2.9. *Let the functions* h_j^*, ψ_j, *and their domains of definition* $D_j, j = -1, 0, 1, \infty$, *be defined as in Definition* 2.7. *Then we have*

$$P_n(z) = \frac{2}{\sqrt{3\psi_{-1}(z)^4 + 1}} e^{n(h_{-1}^*(z)+3z)} \left(1 + O\left(\frac{1}{n}\right)\right) \text{ as } n \to \infty \text{ for } z \in D_{-1}, \tag{2.33}$$

$$Q_n(z) = \frac{2}{\sqrt{3\psi_0(z)^4 + 1}} e^{nh_0^*(z)} \left(1 + O\left(\frac{1}{n}\right)\right) \text{ as } n \to \infty \text{ for } z \in D_0, \tag{2.34}$$

$$R_n(z) = \frac{2}{\sqrt{3\psi_1(z)^4 + 1}} e^{n(h_1^*(z)-3z)} \left(1 + O\left(\frac{1}{n}\right)\right) \text{ as } n \to \infty \text{ for } z \in D_1, \tag{2.35}$$

$$E_n(z) = \frac{2}{\sqrt{3\psi_\infty(z)^4 + 1}} e^{nh_\infty^*(z)} \left(1 + O\left(\frac{1}{n}\right)\right) \text{ as } n \to \infty \text{ for } z \in D_\infty. \tag{2.36}$$

The Landau symbols $O(\cdot)$ *in* (2.33)–(2.36) *hold uniformly on compact subsets of each of the domains of definition of the asymptotic relations. The signs of the square roots are assumed to be positive for real* z.

Remarks. (1) From (2.8) and the definition of ψ by (2.5) and (2.6) we deduce that the equation $3\psi_j(z)^4 + 1 = 0$ has its only solutions at the four branch points $z_1, \ldots, z_4$. From (2.27) it therefore follows that the square roots in (2.33)–(2.36) are analytic and different from zero through out the domains D_{-1}, D_0, D_1, and $D_\infty \setminus \{0\}$.

(2) In order to apply the asymptotic relations of Theorem 2.9, one has to calculate the values $\psi_j(z)$ and $h_j^*(z)$ for a given $z \in D_j, j = -1, 0, 1, \infty$. It has already been mentioned in remarks to Lemma 2.6 and Definition 2.7 that this can be done very efficiently by solving the cubic Eq. (2.11) with $\pi(\zeta)$ replaced by z. The selection of the right solution out of the three candidates depends on the domain D_j. In neighborhoods of $z = 0$ and $z = \infty$, the selection is clear, since the rules follow immediately from conditions (2.24)–(2.26) in Definition 2.7. Throughout each domain D_j the selection can then be done by continuation, i.e., one follows a path through the domain D_j, and at each new value of z one selects the solution of Eq. (2.11) that lies nearest to the previously selected one. Of course, in order that this procedure works, consecutive points of z must not lie too far apart.

(3) Comparing the right-hand sides of the asymptotic relations (2.3) in Theorem 2.2 with the representations (2.31) and (2.32) of the functions $h_j, j = -1, 0, 1, \infty$, in

Theorem 2.8, one sees that the right-hand sides of relations (2.3) can be expressed with the help of the functions h_j. Since $h_j = \operatorname{Re} h_j^*, j = -1, 0, 1$, it is immediate that Theorem 2.2 is a corollary of Theorem 2.9.

2.5. Calculation of the arcs $\Gamma_{\dots}$ and of the measures v_j

While the functions h_j, h_j^*, and $\psi_j, j = -1, 0, 1, \infty$, can be calculated very efficiently by solving Eq. (2.11), we have up to now still not presented an efficient numerical method for the calculation of the density functions of the measures $v_j, j = -1, 0, 1, \infty$, and for the arcs $\Gamma_{-1}, \Gamma_1, \Gamma_{01}, \dots, \Gamma_{04}, \Gamma_{\infty 1}, \dots, \Gamma_{\infty 4}$ that form the supports of these measures and also the boundaries of the domains D_j. In the next two theorems the numerical tools for such calculations are presented.

Theorem 2.10. *Let the sets $K_j, j = -1, 0, 1, \infty$, be defined s in Lemma* 2.6 *with $K_{-1} = \Gamma_{-1}, K_0 = \Gamma_{00} \cup \dots \cup \Gamma_{04}, K_1 = \Gamma_1, K_\infty = \Gamma_{\infty 1} \cup \dots \cup \Gamma_{\infty 4}, \Gamma_{00} = [-iy_1, iy_1]$, and $y_1 \doteq 0.621391$.*

(i) *Out of the eleven Jordan arcs $\Gamma_{-1}, \Gamma_1, \Gamma_{01}, \dots, \Gamma_{04}, \Gamma_{\infty 1}, \dots, \Gamma_{\infty 4}$, the three arcs $\Gamma_1, \Gamma_{01}, \Gamma_{\infty 1}$ start at the branch point z_1. There, these three arcs have the tangential directions*

$$\varphi_1 = 65\pi/36, \quad \varphi_{01} = 41\pi/36, \quad \varphi_{\infty 1} = 17\pi/36, \tag{2.37}$$

respectively. At the three other branch points z_2, z_3, and z_4 one gets corresponding results by symmetry for the three groups of arcs $\{\Gamma_{-1}, \Gamma_{02}, \Gamma_{\infty 2}\}$, $\{\Gamma_{-1}, \Gamma_{03}, \Gamma_{\infty 3}\}$, and $\{\Gamma_1, \Gamma_{04}, \Gamma_{\infty 4}\}$. Hence, by (2.37) *and its symmetric repetitions at the points z_2, z_3, z_4, we have initial directions for all subarcs $\Gamma_{-1}, \dots, \Gamma_{\infty 4}$. The interval $\Gamma_{00} = [-iy_1, iy_1]$ belongs not to this list, however, this poses no problems.*

(ii) *Let $t_z \in \partial\mathbb{D}$ denote the tangent to the arcs $\Gamma_{-1}, \Gamma_1, \Gamma_{01}, \dots, \Gamma_{04}, \Gamma_{\infty 1}, \dots, \Gamma_{\infty 4}$ at the point z. Then we have*

$$t_z = \pm i \frac{\overline{\psi_0(z) - \psi_\infty(z)}}{|\psi_0(z) - \psi_\infty(z)|} \quad \text{for } z \in \Gamma_1 \cup \Gamma_{-1}, \tag{2.38}$$

$$t_z = \pm i \frac{\overline{\psi_1(z) - \psi_\infty(z)}}{|\psi_1(z) - \psi_\infty(z)|} \quad \text{for } z \in \Gamma_{01} \cup \Gamma_{04}, \tag{2.39}$$

$$t_z = \pm i \frac{\overline{\psi_{-1}(z) - \psi_\infty(z)}}{|\psi_{-1}(z) - \psi_\infty(z)|} \quad \text{for } z \in \Gamma_{02} \cup \Gamma_{03}, \tag{2.40}$$

$$t_z = \pm i \frac{\overline{\psi_1(z) - \psi_0(z)}}{|\psi_1(z) - \psi_0(z)|} \quad \text{for } z \in \Gamma_{\infty 1} \cup \Gamma_{\infty 4}, \tag{2.41}$$

$$t_z = \pm i \frac{\overline{\psi_{-1}(z) - \psi_0(z)}}{|\psi_{-1}(z) - \psi 1_0(z)|} \quad \text{for } z \in \Gamma_{\infty 2} \cup \Gamma_{\infty 3} \tag{2.42}$$

with $\psi_j, j = -1, 0, 1, \infty$, *denoting the functions introduced in Definition* 2.7. *At the points* $z = z_1, \dots, z_4$ *expressions* (2.38)–(2.42) *are undetermined, but the missing information is supplied by* (2.37) *and its symmetric repetitions.*

Remark. The initial directions given in (2.37) together with the formulae for the tangent vectors given in (2.38)–(2.42) provide an efficient algorithm for the calculation of the Jordan arcs $\Gamma_{-1}, \Gamma_1, \Gamma_{01}, \dots, \Gamma_{04}, \Gamma_{\infty 1}, \dots, \Gamma_{\infty 4}$ in an obvious way. The calculation should always start at one of the four branch points $z_j, j = 1, \dots, 4$. Near each branch point, it is easy to select the two solutions $\psi_j(z)$ and $\psi_{j'}(z)$ that are relevant in (2.38)–(2.42) out of the three solutions of Eq. (2.11), since near the branch points z_j, the two relevant values are nearly identical and different from the third one. Expressions (2.38)–(2.42) define a differential equation for the numerical calculation of the arcs using the branch points as initial values.

Theorem 2.11. *Let* ds_z *denote the line element at a point* z *of the open subarcs of the sets* $K_j, j = -1, 0, 1, \infty$, *which have been defined as in Theorem* 2.10 *and in Lemma* 2.6. *For the density functions of the measures* $\nu_j, j = -1, 0, 1, \infty$, *introduced in Theorem* 2.8, *we have the representations*

$$\frac{d\nu_j}{ds_z}(z) = \frac{3}{2\pi} |\psi_{j+}(z) - \psi_{j-}(z)| \quad \text{for } z \in K_j, \tag{2.43}$$

where ψ_{j+} *and* ψ_{j-} *are the boundary values of* ψ_j *from both sides of the open subarcs of the sets* K_j.

For each $j \in \{-1, 0, 1, \infty\}$ *the boundary values* ψ_{j+} *and* ψ_{j-} *of* ψ_j *are on each open subarc of* K_j *identical with two other functions* ψ_{l_1} *and* ψ_{l_2}, $l_1, l_2 \in \{-1, 0, 1, \infty\} \setminus \{j\}$. *Considering individually the subarcs* $\Gamma_{-1}, \Gamma_1, \Gamma_{00}, \dots, \Gamma_{04}$, $\Gamma_{\infty 1}, \dots, \Gamma_{\infty 4}$ *of the sets* $K_j, j = -1, 0, 1, \infty$, *as stated in Theorem* 2.10, *one then has the following more specific representations for the density functions* $\nu_j, j = -1, 0, 1, \infty$:

$$\frac{d\nu_{-1}}{ds_z}(z) = \frac{3}{2\pi} |\psi_0(z) - \psi_\infty(z)| \quad \text{for } z \in \Gamma_{-1}, \tag{2.44}$$

$$\frac{d\nu_1}{ds_z}(z) = \frac{3}{2\pi} |\psi_0(z) - \psi_\infty(z)| \quad \text{for } z \in \Gamma_1, \tag{2.45}$$

$$\frac{d\nu_0}{ds_z}(z) = \frac{3}{2\pi} \begin{cases} |\psi_1(z) - \psi_\infty(z)| & \text{for } z \in \Gamma_{01} \cup \Gamma_{04}, \\ |\psi_1(z) - \psi_{-1}(z)| & \text{for } z \in \Gamma_{00} = [-iy_1, iy_1], \\ |\psi_{-1}(z) - \psi_\infty(z)| & \text{for } z \in \Gamma_{02} \cup \Gamma_{03}, \end{cases} \tag{2.46}$$

$$\frac{d\nu_\infty}{ds_z}(z) = \frac{3}{2\pi} \begin{cases} |\psi_1(z) - \psi_0(z)| & \text{for } z \in \Gamma_{\infty 1} \cup \Gamma_{\infty 4}, \\ |\psi_{-1}(z) - \psi_0(z)| & \text{for } z \in \Gamma_{\infty 2} \cup \Gamma_{\infty 3}, \end{cases}. \tag{2.47}$$

Remarks. (1) Expressions (2.44)–(2.47) give nearly explicit representation for the densities of the measures ν_{-1}, ν_0, ν_1, and ν_∞. For their calculation one only needs the values $\psi_i(z), j = -1, 0, 1, \infty$, which are the same as those used for the calculation of the arcs in Theorem 2.10, and they are solutions of Eq. (2.11). It is recommended to calculate the arcs $\Gamma_{j\cdot}$, and the density functions of the measures $\nu_j, j = -1, 0, 1, \infty$, simultaneously.

(2) In Theorem 2.8, representations for the measures $\nu_j, j = -1, 0, 1, \infty$, have already been given by formula (2.30). However these representations contain normal derivatives of the function h, and therefore are not immediately suited for numerical calculations.

3. Proofs, Part I

In the present section we prove results that are concerned with the geometric side of the problem, i.e., we prove the Lemmas 2.5, 2.6, and Theorem 2.8, where the existence and specific properties of the arcs $\Gamma_{j\cdot}$, of the measures ν_j, and the functions $h_j, h_j^*, \psi_j, j = -1, 0, 1, \infty$ have been stated. Further, we prove the Theorems 2.10 and 2.11, which are concerned with the numerical calculation of the arcs $\Gamma_{j\cdot}$ and the measures ν_j. The asymptotic relations themselves will be proved in Section 4.

3.1. *Proof of Lemma 2.5*

First, we shall show that the function h defined by (2.12) and (2.13) has developments (2.14)–(2.17). This can be done by straightforward calculations. Indeed, from (2.6) and (2.5) we deduce that the three branches of the function $\psi \circ \pi^{-1}$ near ∞ have the developments

$$\psi \circ \hat{\pi}_{-1}^{-1}(z) = -1 + \frac{1}{3z} + O\left(\frac{1}{z^2}\right) \quad \text{as } z \to \infty,$$

$$\psi \circ \hat{\pi}_0^{-1}(z) = \frac{1}{3z} + O\left(\frac{1}{z^3}\right) \quad \text{as } z \to \infty,$$

$$\psi \circ \hat{\pi}_1^{-1}(z) = 1 + \frac{1}{3z} + O\left(\frac{1}{z^2}\right) \quad \text{as } z \to \infty \tag{3.1}$$

and near the origin we have

$$\psi \circ \hat{\pi}_{\infty}^{-1}(z) = \frac{1}{3z} + O(z) \quad \text{as } z \to 0 \tag{3.2}$$

for the branch that corresponds to the point $\zeta_0 \in \mathscr{R}$. In (3.1) the $\hat{\pi}_j^{-1}, j = -1, 0, 1$, denote the three branches of π^{-1} at infinity satisfying $\hat{\pi}_j^{-1}(\infty) = \infty^{(j)}, j = -1, 0, 1$, with a numbering of the points $\infty^{(j)} \in \mathscr{R}$ as defined in the lemma. In (3.2) $\hat{\pi}_{\infty}^{-1}$ denotes the branch of π^{-1} at the origin satisfying $\hat{\pi}_{\infty}^{-1}(0) = \zeta_0$. Using (2.13) then yields

$$u\circ\psi\circ\hat{\pi}_{-1}^{-1}(z) = -3z + \log z + O\left(\frac{1}{z}\right) \quad \text{as } z \to \infty,$$
$$u\circ\psi\circ\hat{\pi}_{0}^{-1}(z) = \log z + \log 2 + i\pi + O\left(\frac{1}{z^2}\right) \quad \text{as } z \to \infty,$$
$$u\circ\psi\circ\hat{\pi}_{1}^{-1}(z) = 3z + \log z + O\left(\frac{1}{z}\right) \quad \text{as } z \to \infty,$$
$$u\circ\psi\circ\hat{\pi}_{\infty}^{-1}(z) = 3\log z + O(1) \quad \text{as } z \to 0, \tag{3.3}$$

from which it immediately follows that the function h possesses the developments (2.14)–(2.17).

Let us now assume that $\tilde{h}$ is a function harmonic in $\mathcal{R}\setminus\{\infty^{(-1)}, \infty^{(0)}, \infty^{(1)}, \zeta_0\}$ and satisfying (2.14)–(2.17). Then the difference $g := h - \tilde{h}$ is harmonic throughout $\mathcal{R}$, and from (2.14) it follows that $g(\infty^{(-1)}) = 0$. Since $\mathcal{R}$ is a compact Riemann surface, it follows that $g \equiv 0$, which proves the lemma. □

3.2. *Four useful lemmas*

The next four lemmas pave the way for the proofs of Lemma 2.6 and Theorem 2.8.

For the derivative of the function $u\circ\psi$ there exists a very nice and simple formula, which turns out to be of key importance at several places. We state and prove it in the next lemma.

Lemma 3.1. *Let $\tilde{\pi}^{-1}$ be a local branch of the inverse projection π^{-1} defined in a domain $U \subseteq \mathbb{C}$, then we have*

$$(u\circ\psi\circ\tilde{\pi}^{-1})'(z) = 3(\psi\circ\tilde{\pi}^{-1})(z) = 3\psi(\zeta) \tag{3.4}$$

for $z \in U, \zeta = \tilde{\pi}^{-1}(z) \in \mathcal{R}$, and u defined by (2.13).

Proof. From the chain rule it follows that

$$(u\circ\psi\circ\tilde{\pi}^{-1})'(z) = u'((\psi\circ\tilde{\pi}^{-1})(z))(\psi\circ\tilde{\pi}^{-1})'(z) = \frac{u'(v)}{(\pi\circ\psi^{-1})'(v)} \tag{3.5}$$

with $v = \psi(\zeta)$. If on the right-hand side of (3.5) one uses for $(\pi\circ\psi^{-1})'(v)$ expression (2.8) and the derivative $u'(v) = -(3v^4 + 1)/(v(v^2 - 1)^2)$, it then follows from (3.5) that

$$(u\circ\psi\circ\tilde{\pi}^{-1})'(z) = 3v, \tag{3.6}$$

which proves (3.4). □

In the next lemma we consider properties of arcs $\tilde{\Gamma}$, on which branches of the function h have identical values. We give explicit representations for the difference of the normal derivatives of h to both sides of these arcs. The results are basically local. The global aspects of intersection arcs are studied in the Lemmas 3.3 and 3.5.

Lemma 3.2. *Let $D_1, D_2 \subseteq \mathcal{R}$ be two domains satisfying $\pi(D_1) = \pi(D_2) =: D \subseteq \bar{\mathbb{C}}$, and let the projection π be univalent in each $D_j, j = 1, 2$. We define*

$$\tilde{h}_j := h \circ \tilde{\pi}_j^{-1}, \qquad \tilde{\pi}_j^{-1} := (\pi|_{D_j})^{-1}, \quad j = 1, 2,$$

and

$$\tilde{\Gamma} := \{ z \in D \mid \tilde{h}_1(z) = \tilde{h}_2(z) \}. \tag{3.7}$$

(i) *The set $\tilde{\Gamma}$ consists of disjoint, open, analytic Jordan arcs. Each open subarc $\tilde{\Gamma}_0$ of $\tilde{\Gamma}$ can be extended in $\tilde{\Gamma}$ up to the boundary ∂D.*

(ii) *The complex number $t_z \in \mathbb{T}$ is a tangent vector to a subarc $\tilde{\Gamma}_0 \subseteq \tilde{\Gamma}$ at the point $z \in \tilde{\Gamma}_0$ if, and only if,*

$$\mathrm{Re}[t_z(\psi \circ \tilde{\pi}_1^{-1}(z) - \psi \circ \tilde{\pi}_2^{-1}(z))] = 0. \tag{3.8}$$

(iii) *Let $\tilde{\Gamma}_0$ be an oriented, open subarc of $\tilde{\Gamma}$, and let $\partial/\partial n_+$ and $\partial/\partial n_-$ be the two normal derivatives to the right and left side of $\tilde{\Gamma}_0$, respectively. Then we have*

$$\frac{\partial}{\partial n_+} \tilde{h}_1(z) + \frac{\partial}{\partial n_-} \tilde{h}_2(z) = 3\lambda |\psi \circ \tilde{\pi}_1^{-1}(z) - \psi \circ \tilde{\pi}_2^{-1}(z)| \quad \text{for } z \in \tilde{\Gamma}_0, \tag{3.9}$$

where λ is a constant equal to -1 or 1 for all $z \in \tilde{\Gamma}_0$ on a subarc $\tilde{\Gamma}_0$ of $\tilde{\Gamma}$.

(iv) *We have*

$$\frac{\partial}{\partial n_+} \tilde{h}_1(z) \neq \frac{\partial}{\partial n_+} \tilde{h}_2(z) \quad \text{for all } z \in \tilde{\Gamma} \backslash \{z_1, \ldots, z_4\}. \tag{3.10}$$

Remarks. (1) We recall that the arcs in Fig. 3 are intersections of branches of the function h. In Fig. 3 at the points $z_1, \ldots, z_4, iy_1, -iy_1$ we observe bifurcations. Why is this no contradiction to assertion (i) in the lemma? Indeed, assertion (i) implies that the intersection arcs of two branches of the function h cannot bifurcate in D. However, it follows from the assumption in the lemma that the four branch points $z_1, \ldots, z_4$ cannot lie in D since otherwise the projection π would not be univalent in the domains D_1 and D_2. This settles the question for the points $z_1, \ldots, z_4$. At the two points iy_1 and $-iy_1$ not only two, but three different branches of the function h are involved. Each of the three arcs that meet at these two points in Fig. 3 can be continued across the bifurcation point, but the continuations are invisible in Fig. 3 since they become irrelevant for the problem under investigation.

(2) In relation (3.10) the four branch points $z_1, \ldots, z_4$ have been excluded explicitly. But this was not really necessary since, as has already been explained in the last remark, we have $z_1, \ldots, z_4 \notin D$ because of the assumed univalence of the projection π in D_1 and D_2.

Proof. From Lemma 3.1 we know that

$$(h^*\circ\tilde{\pi}_j^{-1})'(z) = 3\psi\circ\tilde{\pi}_j^{-1}(z) \qquad \text{for all } z\in D,\ j=1,2. \tag{3.11}$$

We define

$$d := \tilde{h}_1 - \tilde{h}_2, \tag{3.12}$$

and deduce from (3.11) that

$$\frac{\partial}{\partial x}d(z) = \mathrm{Re}(h^*\circ\tilde{\pi}_1^{-1} - h^*\circ\tilde{\pi}_2^{-1})'(z) = 3\,\mathrm{Re}[\psi\circ\tilde{\pi}_1^{-1}(z) - \psi\circ\tilde{\pi}_2^{-1}(z)], \tag{3.13}$$

$$\frac{\partial}{\partial y}d(z) = -\mathrm{Im}(h^*\circ\tilde{\pi}_1^{-1} - h^*\circ\tilde{\pi}_2^{-1})'(z) = -3\,\mathrm{Im}[\psi\circ\tilde{\pi}_1^{-1}(z) - \psi\circ\tilde{\pi}_2^{-1}(z)]$$

for $z\in D$.

Since it has been assumed that the projection π is univalent in the two domains D_1 and D_2, it follows that $z_1,\dots,z_4\notin D$. Hence, it follows from (3.13) that $\partial d/\partial x$ and $\partial d/\partial y$ cannot vanish at the same time at any point $z\in D$, or in other words, the function d has no critical points in the domain D. Assertion (i) therefore follows from the Implicit Function Theorem.

Let $\tilde{\Gamma}_0$ be an open, analytic subarc of $\tilde{\Gamma}$. Then $t = t_z\in\mathbb{T}$ is a tangent vector to $\tilde{\Gamma}_0$ at the point $z\in\tilde{\Gamma}_0$ if, and only if,

$$\frac{\partial}{\partial t}d(z) = 0. \tag{3.14}$$

We have

$$\frac{\partial}{\partial t}d(z) = \mathrm{Re}(t_z)\frac{\partial}{\partial x}d(z) + \mathrm{Im}(t_z)\frac{\partial}{\partial y}d(z). \tag{3.15}$$

Hence, relation (3.8) follows from (3.11) and (3.13)–(3.15).

Since the tangent vector t and the two normal vectors $n_\pm$ are orthogonal, it follows from (3.14) and (3.11) that

$$\begin{aligned}\left|\frac{\partial}{\partial n_\pm}d(z)\right| &= \left|\frac{\partial}{\partial z}(h^*\circ\tilde{\pi}_1^{-1} - h^*\circ\tilde{\pi}_2^{-1})\right| \\ &= 3|\psi\circ\tilde{\pi}_1^{-1}(z) - \psi\circ\tilde{\pi}_2^{-1}(z)| \quad \text{for } z\in\tilde{\Gamma}_0.\end{aligned} \tag{3.16}$$

On the other hand, we have

$$\frac{\partial}{\partial n_+}\tilde{h}_j(z) = -\frac{\partial}{\partial n_-}\tilde{h}_j(z) \qquad \text{for } z\in\tilde{\Gamma}_0,\ j=1,2. \tag{3.17}$$

Relations (3.16) and (3.17) together yield that

$$\left|\frac{\partial}{\partial n_+}\tilde{h}_1(z) + \frac{\partial}{\partial n_-}\tilde{h}_2(z)\right| = 3|\psi\circ\tilde{\pi}_1^{-1}(z) - \psi\circ\tilde{\pi}_2^{-1}(z)| \quad \text{for } z\in\tilde{\Gamma}_0, \tag{3.18}$$

and from (3.18) we deduce relation (3.9). Thus, it only remains to show that the factor λ in (3.9) is constant on each subarc $\tilde{\Gamma}_0$ of $\tilde{\Gamma}$.

We know that the function d has no critical point in D. Therefore, it follows from (3.12), (3.14), and $z_1, \dots, z_4 \notin D$ that

$$\frac{\partial}{\partial n_\pm} d(z) = \frac{\partial}{\partial n_\pm}(\tilde{h}_1(z) - \tilde{h}_2(z)) \neq 0 \quad \text{for } z \in \tilde{\Gamma}_0. \tag{3.19}$$

With identity (3.16) we see that $\frac{\partial}{\partial n_+}\tilde{h}_1(z) - \frac{\partial}{\partial n_+}\tilde{h}_2(z) = \frac{\partial}{\partial n_+}\tilde{h}_1(z) + \frac{\partial}{\partial n_-}\tilde{h}_2(z)$ cannot have a sign change on any subarc $\tilde{\Gamma}_0$. This completes the proof of assertion (iii).

Since $\tilde{\Gamma}_0$ is an arbitrary subarc of $\tilde{\Gamma}$, assertion (iv) is an immediate consequence of (3.19). □

In a technical sense the next lemma forms the basis for the proof of Lemma 2.6 and also for the proof of Theorem 2.8.

Lemma 3.3. (i) *There exist four Jordan arcs* $\Gamma_{\infty j}, j = 1, \dots, 4$, *each contained in the corresponding jth quadrant of* $\bar{\mathbb{C}}$ *and connecting the branch point* z_j *with* ∞. *The arcs are uniquely determined by the following two assumptions*:

(α) *Let* $D_1 \subseteq \bar{\mathbb{C}}$ *denote the domain* $\bar{\mathbb{C}} \backslash (\Gamma_{\infty 1} \cup \Gamma_{\infty 4})$ *and* $\hat{\pi}_1^{-1}$ *the branch function of the inverse projection* π^{-1} *in* D_1 *that satisfies*

$$\psi \circ \hat{\pi}_1^{-1}(0) = \sqrt{1/3}. \tag{3.20}$$

It is assumed that the two arcs $\Gamma_{\infty 1}$ *and* $\Gamma_{\infty 4}$ *are chosen in such a way that the function* $\hat{h}_1 := h \circ \hat{\pi}_1^{-1}$ *can continuously be extended throughout* $\mathbb{C}$.

(β) *Let* $D_{-1} \subseteq \bar{\mathbb{C}}$ *denote the domain* $\bar{\mathbb{C}} \backslash (\Gamma_{\infty 2} \cup \Gamma_{\infty 3})$ *and* $\hat{\pi}_{-1}^{-1}$ *the branch function of the inverse projection* π^{-1} *in* D_{-1} *that satisfies*

$$\psi \circ \hat{\pi}_{-1}^{-1}(0) = -\sqrt{1/3}. \tag{3.21}$$

As in assumption (α), *it is again assumed that the two arcs* $\Gamma_{\infty 2}$ *and* $\Gamma_{\infty 3}$ *are chosen in such a way that the function* $\hat{h}_{-1} := h \circ \hat{\pi}_{-1}^{-1}$ *can continuously be extended throughout* $\mathbb{C}$.

(ii) *Let* $D_0 \subseteq \bar{\mathbb{C}}$ *denote the domain* $\bar{\mathbb{C}} \backslash (\Gamma_{\infty 1} \cup \dots \cup \Gamma_{\infty 4})$ *and* $\hat{\pi}_0^{-1}$ *the branch function of the inverse projection* π^{-1} *in* D_0 *that satisfies*

$$\psi \circ \hat{\pi}_0^{-1}(0) = \infty. \tag{3.22}$$

If the two assumptions (α) *and* (β) *are satisfied, then also the function* $\hat{h}_0 := h \circ \hat{\pi}_0^{-1}$ *can continuously be extended throughout* $\mathbb{C} \backslash \{0\}$. *The function* $\hat{h}_0$ *has a logarithmic singularity at the origin.*

(iii) *We have*

$$\operatorname{Re}(z) = \begin{cases} \frac{1}{3}\log 2 + O\left(\frac{1}{|z|}\right) & \text{as } |z| \to \infty \text{ and } z \in \Gamma_{\infty 1} \cup \Gamma_{\infty 4}, \\ -\frac{1}{3}\log 2 + O\left(\frac{1}{|z|}\right) & \text{as } |z| \to \infty \text{ and } z \in \Gamma_{\infty 2} \cup \Gamma_{\infty 3}. \end{cases} \tag{3.23}$$

In a neighborhood D of infinity we have

$$\hat{h}_0(z) = h_{\max}(z) \quad \textit{for } z \in D, \quad \textit{and further}$$

$$\hat{h}_0(z) = \max(-3\,\mathrm{Re}(z), 3\,\mathrm{Re}(z), \log 2) + \log|z| + O\left(\frac{1}{|z|}\right) \quad \textit{as } |z| \to \infty,$$

$$\hat{h}_j(z) = \min(j3\,\mathrm{Re}(z), \log 2) + \log|z| + \mathrm{O}\left(\frac{1}{|z|}\right) \quad \textit{as } |z| \to \infty, \ j = -1, 1, \tag{3.24}$$

with $h_{\max}$ *defined in* (2.18).

Definition 3.4. On the Riemann surface $\mathscr{R}$ three sheets $B_j, j = -1, 0, 1$, are defined in the following way:

(i) Let D_j, and $\hat{\pi}_j^{-1}, j = -1, 0, 1$, be the three domains and the corresponding three branches of the inverse projection π^{-1} in D_j introduced in Lemma 3.3. Then the domains $\hat{\pi}_j^{-1}(D_j) \subseteq \mathscr{R}$ are the inner part of the three sheets $B_j, j = -1, 0, 1$.

(ii) The pairs of domains $(\hat{\pi}_0^{-1}(D_1), \hat{\pi}_1^{-1}(D_0))$ and $(\hat{\pi}_{-1}^{-1}(D_{-1}), \hat{\pi}_0^{-1}(D_0))$ are separated by two Jordan curves $C_{\infty 1}$ and $C_{\infty,-1}$, respectively. Pieces of these curves are attributed to the neighboring sheets B_1, B_0 and B_{-1}, B_0 in such a way that the projection π is univalent on each sheet $B_j, j = -1, 0, 1$. It is immediate that such an attribution is always possible.

Proof of Lemma 3.3. The defining property of the Jordan arcs $\Gamma_{\infty l}, l = 1, \ldots, 4$, is the continuity of the three branch functions $\hat{h}_j, j = -1, 0, 1$, of h across the arcs $\Gamma_{\infty l}$. The function h has been introduced in Definition 2.7 via the v-plane $\bar{\mathbb{C}}$, which can bijectively be mapped onto the Riemann surface $\mathscr{R}$ by ψ. In the proof of existence for the four Jordan arcs $\Gamma_{\infty l}, l = 1, \ldots, 4$, we shall again recur to the v-plane, where we shall define three domains $G_j, j = -1, 0, 1$, which, via the mapping $\pi \circ \psi^{-1}$, correspond to the domains D_j introduced in the lemma.

Near infinity the situation is rather clear, and therefore we shall start with a local consideration near that point. Let the $U_j, j = -1, 0, 1$, be three open neighborhoods of the three points $v = j$ in the v-plane. We define $\tilde{D}_j := \psi^{-1}(U_j) \subseteq \mathscr{R}$, and assume that $\pi(\tilde{D}_j) =: D \subseteq \bar{\mathbb{C}}$ is identical for all three indices $j = -1, 0, 1$. As in (2.31), we deduce from the developments (3.3) that the branch functions $\tilde{h}_j := h \circ \tilde{\pi}_j^{-1}$, $\tilde{\pi}_j^{-1} := (\pi|_{D_j})^{-1}$, associated with the domains $\tilde{D}_j$ have the developments

$$\tilde{h}_j(z) = 3j\,\mathrm{Re}(z) + \log|z| + (1 - j^2)\log 2 + O\left(\frac{1}{|z|}\right) \quad \text{as } |z| \to \infty \ \text{ for } j = -1, 0, 1. \tag{3.25}$$

From (3.25) it follows that near infinity the two arcs

$$\Gamma^j := \{z \in D \mid \tilde{h}_0(z) = \tilde{h}_j(z)\}, \quad j = -1, 1, \tag{3.26}$$

are approximately equal to the two vertical lines $3\,\mathrm{Re}(z) = -\log 2$ and $3\,\mathrm{Re}(z) = \log 2$.

Intersection curves of two branches of the function $h \circ \pi^{-1}$ like the arcs Γ^{-1} and Γ^{1} have already been studied in Lemma 3.2, and from there we know that both arcs can be continued analytically as long as they do not hit one of the four branch points $z_1, \dots, z_4$. We shall prove the existence of such continuations in a constructive way. Besides of the existence of the arcs $\Gamma_{\infty l}, l = 1, \dots, 4$, we also learn from this analysis which arc $\Gamma_{\infty l}$ ends at which branch point $z_{l'}$, $l' \in \{1, \dots, 4\}$.

The function $\psi^{-1} : \bar{\mathbb{C}} \to \mathcal{R}$ is monotonically increasing on the three intervals $I_{-1} := (-1, 0)$, $I_0 := \bar{\mathbb{R}} \backslash [-1, 1], I_1 := (0, 1)$, and it is immediate that each interval $I_j, j = -1, 0, 1$, is mapped bijectively on $\mathbb{R}$ by ψ^{-1}. The points $v_l = \sqrt[4]{-1/3}$, $l = 1, \dots, 4$, are the only zeros of the derivative of ψ^{-1}, as has been shown in (2.8), and at the same time they are critical points of the function $\mathrm{Re}\, u$. The function $\mathrm{Re}\, u$ corresponds in the v-plane to the function h on $\mathcal{R}$ (cf. Definition 2.4). The points $v_l, l = 1, \dots, 4$, are the preimages $\psi^{-1}(\zeta_l)$ of the four branch points $\zeta_l \in \mathcal{R}, l = 1, \dots, 4$.

In the v-plane we define three disjoint domains $G_j, j = -1, 0, 1$, which possess the following four properties:

(a) $I_j \subseteq G_j$ for $j = -1, 0, 1$.

(b) $\pi \circ \psi^{-1}$ is univalent in each domain $G_j, j = -1, 0, 1$.

(c) $\overline{G_{-1} \cup G_0 \cup G_1} = \bar{\mathbb{C}}$.

(d) The set $\gamma = \bigcup_{j=1}^{3} \partial G_j$ consists of analytic arcs. We have $v_l \in \gamma, l = 1, \dots, 4$, and for each $v \in \gamma \backslash \{v_1, \dots, v_4\}$ there exists a corresponding $v' \in \gamma \backslash \{v_1, \dots, v_4\}$ such that $v' \neq v$ and

$$\pi \circ \psi^{-1}(v') = \pi \circ \psi^{-1}(v), \tag{3.27}$$

$$\mathrm{Re}\, u(v') = \mathrm{Re}\, u(v). \tag{3.28}$$

The existence of the three domains G_j will be proved by considering level curves of the function $\mathrm{Re} u$. But before we come to the construction of the domains G_j, we discuss the consequences of their existence. Set

$$\hat{D}_j := \psi^{-1}(G_j) \quad \text{and} \quad \hat{\pi}_j^{-1} := (\pi|_{\hat{D}_j})^{-1} \quad \text{for } j = -1, 0, 1. \tag{3.29}$$

Then it follows from (3.27) and (3.28) in property (d) together with property (c) that the two functions $\hat{h}_j = h \circ \hat{\pi}_j^{-1} = \mathrm{Re}\, u \circ \psi \circ \hat{\pi}_j^{-1}, j = -1, 1$, can continuously be extended throughout $\mathbb{C}$, and the function $\hat{h}_0 = h \circ \hat{\pi}_0^{-1} = \mathrm{Re}\, u \circ \psi \circ \hat{\pi}_0^{-1}$ can continuously be extended throughout $\mathbb{C} \backslash \{0\}$. Assumptions (3.20)–(3.22) in the lemma are immediate consequences of property (a). Thus, after the existence of the domains $G_j, j = -1, 0, 1$, has been proved, it remains only to prove with respect to assertions (i) and (ii) of the lemma that the arcs $\Gamma_{\infty l}, l = 1, \dots, 4$, are uniquely determined and that each arc is contained in a different quadrant of $\bar{\mathbb{C}}$.

Estimate (3.23) in assertion (iii) of the lemma is an immediate consequence of the observations made after (3.26). Relations (3.24) follow from developments (3.25) together with a consideration of the function $\mathrm{Re}\, u$ on $\mathbb{R}$ in neighborhoods of the three points $v = -1, 0$, and 1.

We will now construct the three domains $G_j, j = -1, 0, 1$. For this purpose we consider the level lines

$$L_c := \{ v \in \mathbb{C} \setminus \{-1, 0, 1\} \mid \mathrm{Re}\, u(v) = c\}, \quad c \in \mathbb{R}, \tag{3.30}$$

of the function $\mathrm{Re}\, u$ in the v-plane. Pieces of these lines are plotted in Fig. 4. The symmetry of the definition of the functions $\mathrm{Re}\, u$ and ψ^{-1} with respect to the x- and the y-axis implies that the value $c_0 \in \mathbb{R}$ of $\mathrm{Re}\, u$ is identical at all four critical points v_l, $l = 1, \dots, 4$, and consequently all four points lie on the same level line L_{c_0}, which we will call critical level line. Contrary to all other level lines L_c, $c \neq c_0$, the critical level line L_{c_0} is connected.

Indeed, since the derivative u' has a simple zero at each of the four critical points $v_l, l = 1, \dots, 4$, exactly four subarcs of L_{c_0} meet at each of these four points. The function $\mathrm{Re}\, u$ is monotonic on each of the 6 intervals $(-\infty, -1)$, $(-1, 0)$, $(0, 1)$, $(1, \infty) \subset \mathbb{R}, (-\infty, 0), (0, \infty) \subset i\mathbb{R}$, and therefore each interval contains exactly one intersection point with L_{c_0}. The nature of the singularities of $\mathrm{Re}\, u$ at the two points $v = -1$ and 1 implies that L_{c_0} cuts the real axis $\mathbb{R}$ perpendicularly at each of the two points $v = -1$ and 1. These observations together with the harmonicity of the function $\mathrm{Re}\, u$ imply that L_{c_0} consists of 8 subarcs, each one connecting two of the four critical points $v_l, l = 1, \dots, 4$. The subarcs of L_{c_0} in the first quadrant are shown in Fig. 4.

The open set $\bar{\mathbb{C}} \setminus L_{c_0}$ consists of 6 domains, which we denote by $\hat{G}_0$, $\hat{G}_\infty$, $\hat{G}_{j,i}, j, i = -1, 1$, and assume that the numeration is chosen in such a way that $0 \in \hat{G}_0$, $\infty \in \hat{G}_\infty$, and $j \in \partial \hat{G}_{j.i}$ for $j, i = -1, 1$ (see Fig. 4, which, however, covers only the case $j = 1$). We have $\mathrm{Re}\, u(v) > c_0$ for $v \in \hat{G}_0 \cup \hat{G}_{-1,1} \cup \hat{G}_{1,1}$, and $\mathrm{Re}\, u(v) < c_0$ for $v \in \hat{G}_\infty \cup \hat{G}_{-1,-1} \cup \hat{G}_{1,-1}$.

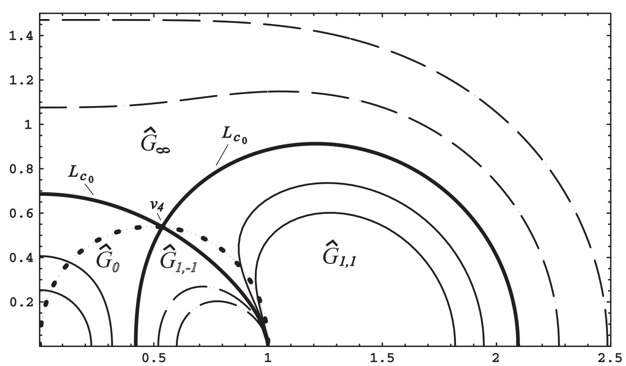

Fig. 4. The critical level line L_{c_0} (thick line), two level lines L_c with $c = c_0 + 0.4$ and $c = c_0 + 0.8$ (full thin lines), two level lines L_c with $c = c_0 - 0.4$ and $c_0 - 0.8$ (dashed thin lines), and the curve $\gamma = \gamma_0 \cup \gamma_1$ (dotted thick line). Only a part of the first quadrant is shown.

The level lines L_c, $c \neq c_0$, consist of three components each, and in every case the three components lie in three different components of $\bar{\mathbb{C}} \backslash L_{c_0}$. For $c < c_0$ the level line L_c consists of the three components L_c^{∞}, L_c^{-1}, and L_c^1 with $L_c^{\infty} \subset \hat{G}_{\infty}$ being a quasi-circle around $v = \infty, L_c^{-1} \subset \hat{G}_{-1,-1}$, and $L_c^1 \subset \hat{G}_{1,-1}$ being punctured quasi-circles going through $v = -1$ and 1, respectively. By definition the points $v = -1$ and 1 should not belong to the two components L_c^{-1} and L_c^1.

For $c > c_0$ the level line L_c consists of the components L_c^{-1}, L_c^0, and L_c^1. Now, $L_c^0 \subset \hat{G}_0$ is a quasi-circle around $v = 0$, while $L_c^{-1} \subset \hat{G}_{-1,1}$, and $L_c^1 \subset \hat{G}_{1,1}$ are again punctured quasi-circles going through $v = -1$ and 1, respectively.

We first concentrate on the construction of the domain G_1, and will construct G_1 as the union of subarcs of the level lines L_c that intersect the interval $(0, 1] = I_1 \cup \{1\}$. The point $v = 1$ has to be added to the open interval $I_1 = (0, 1)$ since at $v = 1$ a whole family of subarcs of the level lines $L_c, c > c_0$, intersects $\mathbb{R}$ perpendicularly. Only a piece of each of these subarcs belongs to the domain G_1.

Let U_0 and U_1 be the neighborhoods introduced before (3.25), and let Γ^1 be the arc introduced in (3.26). As before (3.25), we set $\tilde{\pi}_j^{-1} := (\pi|_{\hat{D}_j})^{-1}$ for $j = 0, 1$, and define the two analytic arcs $\tilde{\gamma}_j := \psi \circ \tilde{\pi}_j^{-1}(\Gamma^1), j = 0, 1$, which are contained in $U_0 \cap \hat{G}_0$ and $U_1 \cap (\hat{G}_{1,1} \cup \{1\})$, respectively. Indeed, this follows from the definition of $U_j, j = 0, 1$, from (3.26), and the behavior of the function $\pi \circ \psi^{-1}$ near the two points $v = 0$ and 1.

From the behavior of the function $\operatorname{Re} u$ in the neighborhoods of the two points $v = 0$ and 1, it further follows that for $c > 0$ near infinity, there exist two points $v_{c,0} \in \tilde{\gamma}_0$, $v_{c,1} \in \tilde{\gamma}_1$, $\operatorname{Im}(v_{c,j}) > 0, j = 0, 1$, such that

$$\pi \circ \psi^{-1}(v_{c,0}) = \pi \circ \psi^{-1}(v_{c,1}) \in \Gamma^1, \tag{3.31}$$

$$\pi \circ \psi^{-1}(\overline{v_{c,0}}) = \pi \circ \psi^{-1}(\overline{v_{c,1}}) \in \Gamma^1, \tag{3.32}$$

$$\operatorname{Re} u(v_{c,0}) = \operatorname{Re} u(v_{c,1}) = c, \tag{3.33}$$

$$\operatorname{Re} u(\overline{v_{c,0}}) = \operatorname{Re} u(\overline{v_{c,1}}) = c. \tag{3.34}$$

Let $\tilde{L}_c^0 \subset L_c^0$ denote the open subarcs of L_c^0 that connects the point $v_{c,0}$ with $\overline{v_{c,0}}$ and intersects $I_1 = (0, 1)$, and let further $\tilde{L}_c^1 \subset L_c^1$ denote the union of the two open, disjoint subarcs of L_c^1 that connect the two points $v_{c,1}$ and $\overline{v_{c,1}}$ with the point $v = 1$. From the behavior of the two functions $\tilde{h}_0$ and $\tilde{h}_1$ near infinity, which we know from (3.25), we can conclude that the two arcs $\pi \circ \psi^{-1}(\tilde{L}_c^0)$ and $\pi \circ \psi^{-1}(\tilde{L}_c^1)$ are disjoined, and further the closure $\overline{\pi \circ \psi^{-1}(\tilde{L}_c^0) \cup \pi \circ \psi^{-1}(\tilde{L}_c^1)}$ forms a Jordan curve $\tilde{\Lambda}_c$ in $\bar{\mathbb{C}}$. It follows from (3.31)–(3.34) and the definition of the $\tilde{L}_c^0$ and $\tilde{L}_c^1$ that the restrictions $\tilde{h}_0|_{\pi \circ \psi^{-1}(\tilde{L}_c^0)}$ and $\tilde{h}_1|_{\pi \circ \psi^{-1}(\tilde{L}_c^1)}$ can be continuously extended to the curve $\tilde{\Lambda}_c$.

If the value $c \in \mathbb{R}$ decreases, then the two points v_{c1} and $\overline{v_{c1}}$ move on $\tilde{\gamma}_1$ away from the point $v = 1$, and the two points v_{c0} and $\overline{v_{c0}}$ move on $\tilde{\gamma}_0$ away from the point $v = 0$. We will have a closer look in this process.

The function $\operatorname{Re} u$ has no other critical points in $\mathbb{C}$ than the four points $v_l, l = 1, \ldots, 4$. With the same arguments as used in Lemma 3.2 and its proof, it follows that the analytic arcs $\tilde{\gamma}_j, j = 0, 1$, can be extended without bifurcations as long as they do not hit a critical point. Especially, the value c can be decreased as long as $c > c_0$. During this process the four points $v_{c0}, \overline{v_{c0}}, v_{c1}, \overline{v_{c1}}$ lie on the prolongations of the two arcs $\tilde{\gamma}_0$ and $\tilde{\gamma}_1$, and the two domains $G_{0c} := \bigcup_{c'>c} \tilde{L}^0_{c'}$ and $G_{c1} := \bigcup_{c'>c} \tilde{L}^1_{c'}$ grow monotonically.

In the limiting situation, when $c \searrow c_0$, then the subarc $\tilde{L}^0_c$ converges to an open subarc $\tilde{L}^0_{c_0}$ of the critical level line L_{c_0}. This subarc $\tilde{L}^0_{c_0}$ connects the two critical points v_1, v_4 and intersects $I_1 = (0, 1)$. The two subarcs of $\tilde{L}^1_c$ also converge to the union $\tilde{L}^1_{c_0}$ of two open subarcs of the critical level line L_{c_0}. This union consists of the two open subarcs that connect the two critical points v_1 and v_4 with the point $v = 1$.

Let us now consider the case $c < c_0$. Here, the situation is simpler, we have only to use the component L^1_c of the level line L_c. This component is contained in the domain $\hat{G}_{1,-1}$, which has $\overline{\tilde{L}^0_{c_0} \cup \tilde{L}^1_{c_0}}$ as boundary, and in the limiting situation $c \nearrow c_0$ the arc L^1_c converges to the arc $\overline{\tilde{L}^0_{c_0} \cup \tilde{L}^1_{c_0}} \setminus \{1\}$.

Finally, the domain G_1 is defined as

$$G_1 := \bigcup_{c<c_0} L^1_c \cup \bigcup_{c \geqslant c_0} (\tilde{L}^0_c \cup \tilde{L}^1_c). \tag{3.35}$$

It is immediate that G_1 is a domain and that $I_1 \subseteq G_1$. Let γ_j, $j = 0, 1$, denote the maximal extensions of the two arcs $\tilde{\gamma}_j$. It follows from the definitions of the arcs L^1_c, $\tilde{L}^0_c$, and $\tilde{L}^1_c$ that

$$\partial G_1 = \gamma_0 \cup \gamma_1. \tag{3.36}$$

For each $c > c_0$, there exists a one-to-one correspondence between the two points v_{c0}, $\overline{v_{c0}}$ and the two points v_{c1}, $\overline{v_{c1}}$, respectively. This correspondence satisfies the relations (3.31)–(3.34), and consequently property (d) is satisfied by the domain G_1. The construction of the subarcs $L^1_c, \tilde{L}^0_c, \tilde{L}^1_c$, and the arcs γ_0 and γ_1 implies that also properties (a) and (b) are satisfied by the domain G_1.

It follows from (3.33) and (3.34) that

$$\pi \circ \psi^{-1}(\gamma_0) = \pi \circ \psi^{-1}(\gamma_1). \tag{3.37}$$

Since $\operatorname{Re}(v_1), \operatorname{Re}(v_4) \in I_1, \operatorname{Im}(v_1) < 0$, and $\operatorname{Im}(v_4) > 0$, the two arcs

$$\Gamma_{\infty 1} := \pi \circ \psi^{-1}(\gamma_0|_{\{\operatorname{Im}(v)<0\}}) \quad \text{and} \quad \Gamma_{\infty 4} := \pi \circ \psi^{-1}(\gamma_0|_{\{\operatorname{Im}(v)>0\}}) \tag{3.38}$$

connect the two branch points $z_1 = \pi \circ \psi^{-1}(v_1)$ and $z_4 = \pi \circ \psi^{-1}(v_4)$ with infinity. Thus, the properties (a), (b), and (d) are proved for the domain G_1.

The domain G_{-1} is defined as the symmetric image of G_1 with respect to the imaginary axis. The role of the points v_1 and v_4 is taken over by v_2 and v_3, and the index value $j = 1$ has to be changed to $j = -1$ everywhere. Correspondingly, the properties (a), (b), and (d) are also true for the domain G_{-1}. Since

$\mathrm{Re}(v_2), \mathrm{Re}(v_3) \in I_{-1}, \mathrm{Im}(v_2) < 0$, and $\mathrm{Im}(v_3) > 0$, the two arcs

$$\Gamma_{\infty 2} := \pi \circ \psi^{-1}(\partial G_{-1} \cap \{\mathrm{Im}(v) < 0\})$$

and

$$\Gamma_{\infty 3} := \pi \circ \psi^{-1}(\partial G_{-1} \cap \{\mathrm{Im}(v) > 0\}) \tag{3.39}$$

connect the two branch points $z_2 = \pi \circ \psi^{-1}(v_2)$ and $z_3 = \pi \circ \psi^{-1}(v_3)$ with infinity.

It will be shown below that $G_1 \cap G_{-1} = \emptyset$. Having this result in mind, we define

$$G_0 := \bar{\mathbb{C}} \backslash (G_1 \cup G_{-1}). \tag{3.40}$$

The properties (a), (b), and (d) carry over from the domains G_1 and G_{-1} to the domain G_0. Property (c) follows immediately from (3.40). Thus, it remains only to show that the two domains G_1 and G_{-1} are, indeed, disjoint.

It follows from the developments in (3.25) that Γ^1 is the only arc in a neighborhood $D \subseteq \bar{\mathbb{C}}$ of infinity on which the function $\hat{h}_1$ has identical values from both sides. (We assume here that $\hat{h}_1$ is initially defined on $\mathbb{R}$.) In the neighborhood D of infinity, we have $\Gamma^1 = (\Gamma_{\infty 1} \cup \Gamma_{\infty 4}) \cap D$. Thus, the arcs $\Gamma_{\infty 1}$ and $\Gamma_{\infty 4}$ are uniquely determined in D. From Lemma 3.2 we then know that the continuations of the two arcs $\Gamma_{\infty 1}$ and $\Gamma_{\infty 4}$ are without bifurcation until a branch point is hit for the first time. Since the continuations are analytic arcs, the uniqueness of the complete arcs $\Gamma_{\infty 1}$ and $\Gamma_{\infty 4}$ is proved. The uniqueness of the two arcs $\Gamma_{\infty 2}$ and $\Gamma_{\infty 3}$ introduced in (3.39) is a consequence of the symmetry.

In order to prove $G_1 \cap G_{-1} = \emptyset$, we consider the difference $d := \hat{h}_0 - \hat{h}_1$ on the imaginary axis $i\mathbb{R}$. From the developments (3.25) we learn that $d(\pm i\infty) = \log 2$, and from (2.17) in Lemma 2.5 that $d(0) = -\infty$. Below we shall show that

$$\frac{\partial}{\partial y} d(iy) > 0 \quad \text{for } y > 0, \ \ y \in \mathbb{R}. \tag{3.41}$$

From (3.41) and the values of d at 0 and ∞, it follows that the difference d has exactly one zero on the positive imaginary half-axis $i\mathbb{R}_+$. By symmetry, there exists another zero on the negative imaginary half-axis $i\mathbb{R}_-$.

If the arc $\Gamma_{\infty 1} \backslash \{\infty\}$ were not contained in the first quadrant $\{z \in \mathbb{C} \mid \mathrm{Re}(z) > 0, \mathrm{Im}(z) > 0\}$, then the difference d would have more than one zero on $i\mathbb{R}_+$, i.e., several different zeros or at least one double zero. Since $d(z) = 0$ for all $z \in \Gamma_{\infty 1} \backslash \{\infty\}$, such a case would contradict the strong monotonicity (3.41). Consequently, it is impossible, it is proved that $\Gamma_{\infty 1} \backslash \{\infty\}$ is contained in the first quadrant. By symmetry the analogous conclusions hold true for the three other arcs $\Gamma_{\infty l}, l = 2, 3, 4$. As a consequence, we see that $G_1 \cap G_{-1} = \emptyset$.

In order to prove (3.41) we deduce as in (3.13) from Lemma 3.1 that

$$\begin{aligned} \frac{\partial}{\partial y} d(iy) &= \frac{\partial}{\partial y}(\hat{h}_0(iy) - \hat{h}_1(iy)) \\ &= -\mathrm{Im}\left(\frac{\partial}{\partial z}(h^* \circ \hat{\pi}_0^{-1} - h^* \circ \hat{\pi}_1^{-1})(iy)\right) = -3\,\mathrm{Im}(\hat{v}_0 - \hat{v}_1) \end{aligned} \tag{3.42}$$

with point $\hat{v}_0, \hat{v}_1 \in \mathbb{C}$ defined by $\hat{v}_j := \psi \circ \hat{\pi}_j^{-1}(iy), j = -1, 0, 1$. Let us assume $y > 0$. Then we have $\hat{v}_0 \in i\mathbb{R}_-$ and $\mathrm{Im}(\hat{v}_j) < 0$ for $j = -1, 1$. From the definition of the $\hat{v}_j$ and from (2.5) and (2.6) we know that

$$0 \equiv \pi \circ \psi^{-1}(\hat{v}_0) - \pi \circ \psi^{-1}(\hat{v}_1) = \frac{\hat{v}_0^2 - 1/3}{\hat{v}_0(\hat{v}_0^2 - 1)} - \frac{\hat{v}_1^2 - 1/3}{\hat{v}_1(\hat{v}_1^2 - 1)} \tag{3.43}$$

or

$$(\hat{v}_0^2 - 1/3)\hat{v}_1(\hat{v}_1^2 - 1) - (\hat{v}_1^2 - 1/3)\hat{v}_0(v_0^2 - 1) = 0. \tag{3.44}$$

Polynomial (3.44) can be divided by the linear factor $(\hat{v}_1 - \hat{v}_0)$, which yields

$$(1 - \hat{v}_0^2) + 2\hat{v}_0\hat{v}_1 - (1 - 3\hat{v}_0^2)\hat{v}_1^2 = 0. \tag{3.45}$$

With the parametrization $\hat{v}_0 = it$, we then have

$$\hat{v}_1 = \hat{v}_1(it) = \frac{it}{1 + 3t^2} + \frac{1}{1 + 3t^2}\sqrt{1 + 3t^2 + 3t^4}. \tag{3.46}$$

From (3.46) and (3.42) we then deduce that

$$\frac{\partial}{\partial y} d(iy) = -3\left(t - \frac{t}{1 + 3t^2}\right) = \frac{-9t^3}{1 + 3t^2} > 0 \quad \text{for } y > 0. \tag{3.47}$$

The last inequality holds since we have $\mathrm{Im}(\hat{v}_0) = t < 0$ for $y > 0$. With the verification of inequality (3.41) the proof of Lemma 3.3 is complete. □

In the next lemma we introduce arcs which will be building blocks for the system of arcs appearing in Lemma 2.6.

Lemma 3.5. (i) *Let $\hat{h}_j$ and $\hat{\pi}_j^{-1}, j = -1, 0, 1$, be the functions introduced in Lemma* 3.3. *The set*

$$N_{01} := \{z \in \bar{\mathbb{C}} \,|\, \hat{h}_0(z) = \hat{h}_1(z)\} \tag{3.48}$$

consists of three analytic arcs, which we denote by $\Gamma_{1k}, k = 1, 2, 3$. Each of these three arcs connects the two branch points z_1 and z_4 in $\bar{\mathbb{C}}$. The arcs are disjoint in $\bar{\mathbb{C}} \backslash \{z_1, z_4\}$. The arc Γ_{11} intersects $\bar{\mathbb{R}}$ in $(0, \infty)$. The arc Γ_{12} intersects $\bar{\mathbb{R}}$ in $(-\infty, 0)$, *and it has two intersection points $\{-iy_1, iy_1\}, y_1 > 0$, with $i\bar{\mathbb{R}}$. The third arc Γ_{13} intersects $\bar{\mathbb{R}}$ at infinity. There are no other intersection points with $\bar{\mathbb{R}}$ or $i\bar{\mathbb{R}}$, and we have $\Gamma_{13} = \Gamma_{\infty 1} \cup \Gamma_{\infty 4}$, where $\Gamma_{\infty 1}$ and $\Gamma_{\infty 4}$ are the arcs that have been introduced in Lemma* 3.3.

(ii) *At the branch point z_1 the three arcs $\Gamma_{1k}, k = 1, 2, 3$, have tangential directions*

$$\varphi_1 = 65\pi/36, \quad \varphi_2 = 41\pi/36, \quad \varphi_3 = 17\pi/36, \tag{3.49}$$

respectively. At the branch point z_4 a symmetric result holds true.

(iii) *The open set $\bar{\mathbb{C}} \backslash N_{01}$ consists of three components, which are denoted by $D_1^{(j)}, j = -1, 0, 1$. The domain $D_1^{(0)}$ contains the origin, the domain $D_1^{(-1)}$ lies to the left of N_{01} and the domain $D_1^{(1)}$ to the right of N_{01}. We have*

$$\hat{h}_0(z) < \hat{h}_1(z) \quad \textit{for } z \in D_1^{(0)}, \tag{3.50}$$

$$\hat{h}_0(z) > \hat{h}_1(z) \quad \textit{for } z \in D_1^{(-1)} \cup D_1^{(1)} \tag{3.51}$$

and

$$\hat{h}_{-1}(z) < \hat{h}_1(z) \quad \textit{for } \operatorname{Re}(z) > 0. \tag{3.52}$$

Remarks. (1) Besides of the set N_{01} we shall also use the set $N_{0,-1} := \{z \in \bar{\mathbb{C}} \mid \hat{h}_0(z) = \hat{h}_{-1}(z)\}$ in the proof of Lemma 2.6. Since $N_{0,-1}$ is the symmetric image of N_{01} with respect to the imaginary axis, all definitions made in Lemma 3.5 can be carried over to the symmetric situation.

(2) The three arcs Γ_{1k}, $k = 1, 2, 3$, are plotted in Fig. 5.

Proof. We first prove inequality (3.52). Set $H_+ := \{z \in \mathbb{C} \mid \operatorname{Re}(z) > 0\}$. From (3.24) in Lemma 3.3 together with developments (3.25) in the proof of Lemma 3.3, we deduce that the function $\hat{h}_1$ is superharmonic in a neighborhood of infinity. From assertion (iii) in Lemma 3.2, we then deduce that the function $\hat{h}_1$ remains superharmonic along the whole arc $\Gamma_{\infty 1} \cup \Gamma_{\infty 4}$, i.e., the sign of λ in (3.9) cannot change on $\Gamma_{\infty 1} \cup \Gamma_{\infty 4}$. As a consequence we see that $\hat{h}_1$ is superharmonic in $\mathbb{C}$. On the other hand the function $\hat{h}_{-1}$ is harmonic in H_+. From the symmetry between the two functions $\hat{h}_1$ and $\hat{h}_{-1}$ it follows that

$$\hat{h}_1(z) = \hat{h}_{-1}(z) \quad \text{for } z \in i\mathbb{R}. \tag{3.53}$$

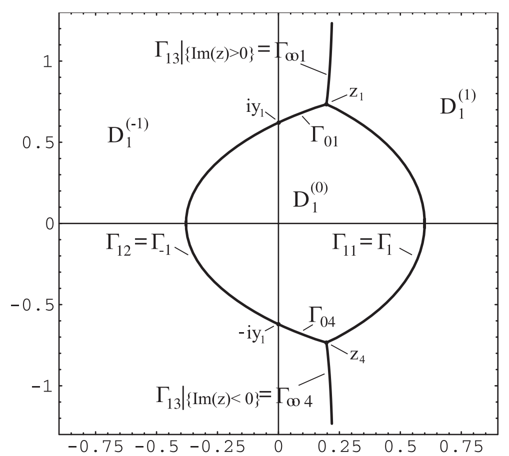

Fig. 5. The three arcs $\Gamma_{11}(=\Gamma_1)$, $\Gamma_{12}(=\Gamma_{-1})$, and $\Gamma_{13}(=\Gamma_{\infty 1} \cup \Gamma_{\infty 4})$ of the set N_{01}, defined in Lemma 3.5, together with three domains $D_1^{(-1)}, D_1^{(0)}$, and $D_1^{(1)}$. (The arc Γ_{13} and the two domains $D_1^{(-1)}$ and $D_1^{(1)}$ are only partially shown. The two arcs Γ_{01} and Γ_{04} are the subarcs of Γ_{12} that connect the point iy_1 with z_1 and the point $-iy_1$ with z_4, respectively.)

Let D be a neighborhood of infinity. From the third line in (3.24) of Lemma 3.3 we deduce that

$$\hat{h}_1(z) \geqslant \hat{h}_{-1}(z) \quad \text{for } z \in D \cap H_+. \tag{3.54}$$

The superharmonicity and the harmonicity of $\hat{h}_1$ and $\hat{h}_{-1}$ together with (3.53) and (3.54) imply inequality (3.52).

From the properties of the two functions $\hat{h}_0$ and $\hat{h}_1$ established in Lemma 3.3 it follows that the difference $d := \hat{h}_0 - \hat{h}_{-1}$ is harmonic in $\mathbb{C} \backslash (\Gamma_{\infty 1} \cup \cdots \cup \Gamma_{\infty 4} \cup \{0\})$. From Lemma 3.3 we further know that

$$d(z) = 0 \quad \text{for } z \in \Gamma_{\infty 1} \cup \Gamma_{\infty 4}. \tag{3.55}$$

Next we show that

$$d(z) > 0 \quad \text{for } z \in \Gamma_{\infty 2} \cup \Gamma_{\infty 3}. \tag{3.56}$$

Indeed, it follows from Lemma 3.3 that $\hat{h}_0(z) = \hat{h}_{-1}(z)$ for $z \in \Gamma_{\infty 2} \cup \Gamma_{\infty 3}$. Because of the symmetry between $\hat{h}_1$ and $\hat{h}_{-1}$ with respect to the imaginary axis, we have $\hat{h}_1(z) - \hat{h}_{-1}(z) = \hat{h}_{-1}(-\bar{z}) - \hat{h}_1(-\bar{z})$ for $z \in \mathbb{C}$. Inequality (3.56) then follows from (3.52).

From (2.17) in Lemma 2.5 we know that $\hat{h}_0(0) = -\infty$. Since $\hat{h}_1$ is bounded at the origin, it follows that

$$d(0) = -\infty. \tag{3.57}$$

We define $D_1^{(0)} := \{z \in \mathbb{C} \mid d(z) < 0\}$. Because of (3.57) we have $0 \in D_1^{(0)}$. Since d is harmonic in $\mathbb{C} \backslash (\Gamma_{\infty 1} \cup \cdots \cup \Gamma_{\infty 4} \cup \{0\})$ and subharmonic at $z = 0$, from (3.55) and (3.56) it follows that the domain $D_1^{(0)}$ is simply connected and $\partial D_1^{(0)}$ is a Jordan curve. From (3.55), (3.56), and the fact that the arcs $\Gamma_{\infty 1}, \ldots, \Gamma_{\infty 4}$ cannot bifurcate, it further follows that $z_1, z_4 \in \partial D_1^{(0)}$ and $z_2, z_3 \notin \partial D_1^{(0)}$. Let Γ_{11} and Γ_{12} denote the two subarcs of $\partial D_1^{(0)}$ that connect the two points z_1 and z_4 in $\partial D_1^{(0)}$.

It has been shown by (3.41) in the proof of Lemma 3.3 that the function d is monotonic on each of the two imaginary half-axis $i\mathbb{R}_+$ and $i\mathbb{R}_-$. Consequently, $\partial D_1^{(0)}$ has exactly one intersection point on each half-axis, we denote these two points by $-iy_1$ and iy_1, $y_1 > 0$. In an analogous way it can be shown that the function d is monotonic on each of the two half-axis $\mathbb{R}_+$ and $\mathbb{R}_-$. The proof is even simpler since in the new situation the two points $\hat{v}_0$ and $\hat{v}_1$ in formula (3.42) are both real. From the monotonicity of d on $\mathbb{R}_+$ and $\mathbb{R}_-$ it follows that $\partial D_1^{(0)}$ has exactly one intersection point on each half-axis. It is geometrically immediate that one of the two arcs Γ_{11} and Γ_{12} intersects $i\mathbb{R}$ in $-iy_1$ and iy_1 and at the same time $\mathbb{R}$ in $(-\infty, 0)$, we denote this arc by Γ_{12}. The other one is denoted by Γ_{11}, and it intersects $\mathbb{R}$ in $(0, \infty)$.

From (3.55) together with what we have shown so far the assertions of part (i) and inequalities (3.50) and (3.51) follow. Thus, it remains only to show that (3.49) holds true.

The two functions $\hat{h}_0$ and $\hat{h}_1$ share the branch point at z_1. For both functions we have the development

$$\hat{h}_j(z) = \tfrac{1}{4}\left(2 + \log\tfrac{1}{3}\right) + 3^{3/4}\,\mathrm{Re}(e^{-\pi/4}(z - z_1)) + \mathrm{Re}(c_1(z - z_1)^{3/2}) + O(|z - z_1|^2) \quad \text{as } z \to z_1, \tag{3.58}$$

$j = 0, 1$, with the coefficient c_1 in the third term given by

$$c_1 := \pm\sqrt{8}3^{-3/8}e^{-i/24}. \tag{3.59}$$

The two signs in (3.59) distinguish between $\hat{h}_0$ and $\hat{h}_1$. We have $h = \mathrm{Re}\,h^*$ and $\hat{h}_j = h\circ\hat{\pi}_j^{-1} = \mathrm{Re}\,h^*\circ\hat{\pi}_j^{-1}, j = 0, 1$. In order to prove (3.58) we derive from (3.4) in Lemma 3.1 that

$$(h^*\circ\hat{\pi}_j^{-1})'(z) = 3\psi\circ\pi_j^{-1}(z) = 3v_1 + \frac{c_1}{2}(z - z_1)^{1/2} + O(z - z_1) \quad \text{as } z \to z_1 \tag{3.60}$$

with $v_1 = 3^{1/4}e^{-i\pi/4}$ and the constant c_1 given in (3.59). The last equality in (3.60), and especially expression (3.59) follow from (2.5) by straightforward calculations. The development (3.58) then follows from (3.60) by integration.

Development (3.58) can be rewritten as

$$\hat{h}_j(z_1 + re^{i\varphi}) = \frac{1}{4}\left(2 + \log\frac{1}{3}\right) + 3^{3/4}r\cos(\varphi - \pi/4) \pm\sqrt{8}3^{-3/8}r^{3/2}\cos\left(\frac{3}{2}\varphi - \frac{5\pi}{24}\right) + O(r^2) \quad \text{as } r \to 0, \tag{3.61}$$

$j = 0, 1$. From definition (3.48) it follows that along the tangential directions of the three subarcs Γ_{1k}, $k = 1, 2, 3$, the second cosine term in (3.61) has to vanish. Hence, for the tangential angles φ_k, $k = 1, 2, 3$, we have

$$\cos\left(\frac{3}{2}\varphi_k - \frac{5\pi}{24}\right) = 0 \quad \text{for } k = 1, 2, 3, \tag{3.62}$$

which implies that

$$\varphi_k = \frac{5\pi}{36} + \frac{\pi}{3} + \frac{2k}{3}\pi \quad \text{for } k = 1, 2, 3, \tag{3.63}$$

and this proves (3.49). Note that the association of individual angles $\varphi_k, k = 1, 2, 3$, with the three arcs Γ_{1k} follows from the global structure of the arcs shown in Fig. 4. □

3.3. Proof of Lemma 2.6

The existence and certain properties of the subarcs of the set Γ are proved in a constructive way by showing the connections between the function $h_{\max}$ defined in (2.18) and the branch functions $\hat{h}_j, j = -1, 0, 1$, of $h\circ\pi^{-1}$ which have been defined in

Lemma 3.3 and are associated with the sheets $B_j, j = -1, 0, 1$, introduced in Definition 3.4.

Besides of the set N_{01} of (3.48) we also use the set $N_{0,-1} = \{z \in \bar{\mathbb{C}} \mid \hat{h}_0(z) = \hat{h}_{-1}(z)\}$, which has already been mentioned in Remark 1 to Lemma 3.5. This set is the symmetric image of N_{01} with respect to reflection on the imaginary axis. Like N_{01}, it consists of three Jordan arcs, which we denote by $\Gamma_{-1,k}, k = 1, 2, 3$, and which connect the two branch points z_2 and z_3 in $\bar{\mathbb{C}}$. The three components of $\mathbb{C}\backslash N_{0,-1}$ are denoted by $D_{-1}^{(j)}, j = -1, 0, 1$, and the numeration is taken in such a way that under reflection on the imaginary axis the $\Gamma_{-1,1}, \Gamma_{-1,2}, \Gamma_{-1,3}$ correspond to $\Gamma_{11}, \Gamma_{12}, \Gamma_{13}$, and the $D_{-1}^{(-1)}, D_{-1}^{(0)}, D_{-1}^{(1)}$ to $D_1^{(1)}, D_1^{(0)}, D_1^{(-1)}$, respectively. We further define $H_+ := \{z \in \mathbb{C} \mid \mathrm{Re}(z) > 0\}$ and $H_- := \{z \in \mathbb{C} \mid \mathrm{Re}(z) < 0\}$.

From (3.50)–(3.52) in Lemma 3.5 we deduce that in the half-plane H_+ we have

$$h_{\max}(z) = \begin{cases} \hat{h}_1(z) & \text{for } z \in D_1^{(0)} \cap H_+, \\ \hat{h}_0(z) & \text{for } z \in D_1^{(1)} \cup (D_1^{(-1)} \cap H_+). \end{cases} \tag{3.64}$$

Because of the symmetry properties with respect to the imaginary axis, in H_- we have

$$h_{\max}(z) = \begin{cases} \hat{h}_{-1}(z) & \text{for } z \in D_1^{(0)} \cap H_-, \\ \hat{h}_0(z) & \text{for } z \in D_{-1}^{(-1)} \cup (D_{-1}^{(1)} \cap H_-). \end{cases} \tag{3.65}$$

From Lemma 3.5 and the symmetry properties we learn that $i\mathbb{R} \cap D_{-1}^{(0)} = i\mathbb{R} \cap D_1^{(0)} = [-iy_1, iy_1]$. Because of symmetry we also have $\hat{h}_1(z) = \hat{h}_{-1}(z)$ for $z \in i\mathbb{R}$. From (3.52) in Lemma 3.5 it then follows that the function $h_{\max}$ is equal to $\hat{h}_1$ on the right side of $[-iy_1, iy_1]$, and equal to $\hat{h}_{-1}$ on the left side of $[-iy_1, iy_1]$. Hence, the function $h_{\max}$ is not harmonic in any neighborhood of any point of $[-iy_1, iy_1]$. The situation is different in $iR\backslash[-iy_1, iy_1]$, where $h_{\max}$ is represented by $\hat{h}_0$ on both sides. From (3.64) and (3.65) we therefore deduce that

$$h_{\max}(z) = \begin{cases} \hat{h}_1(z) & \text{for } z \in D_1^{(0)} \cap H_+, \\ \hat{h}_0(z) & \text{for } z \in \mathbb{C}\backslash\overline{D_1^{(0)} \cup D_{-1}^{(0)}}, \\ \hat{h}_{-1}(z) & \text{for } z \in D_{-1}^{(0)} \cap H_-. \end{cases} \tag{3.66}$$

From Lemma 3.3 we know that $\hat{h}_0$ is harmonic in $\mathbb{C}\backslash(\Gamma_{\infty 1} \cup \cdots \cup \Gamma_{\infty 4} \cup \{0\})$, and it is not harmonic in any neighborhood of a point $z \in \Gamma_{\infty 1} \cup \cdots \cup \Gamma_{\infty 4} \cup \{0\}$. In (3.66) the singularity of $\hat{h}_0$ at the origin $z = 0$ plays no role since there the two functions $\hat{h}_1$ and $\hat{h}_{-1}$ are larger than $\hat{h}_0$. The open arcs $\Gamma_{\infty l}\backslash\{z_l, \infty\}, l = 1, \cdots, 4$, are, however, fully contained in $\mathbb{C}\backslash\overline{D_1^{(0)} \cup D_{-1}^{(0)}}$. From (3.66) we therefore deduce that $h_{\max}$ is not harmonic in any neighborhood of points $z \in \Gamma_{\infty 1} \cup \cdots \cup \Gamma_{\infty 4}$. We define

$$K_\infty := \Gamma_{\infty 1} \cup \cdots \cup \Gamma_{\infty 4}. \tag{3.67}$$

Since $\hat{h}_0$ is different from $\hat{h}_1$ in $D_1^{(0)} \cap H_+$, and different from $\hat{h}_{-1}$ in $D_{-1}^{(0)} \cap H_-$, it follows from (3.66) that $h_{\max}$ is not harmonic in any neighborhood of points $z \in (\partial D_1^{(0)} \cap H_+) \cup (\partial D_{-1}^{(0)} \cap H_-)$. The set Γ in Lemma 2.6 can therefore be represented as

$$\Gamma = K_\infty \cup \partial(D_1^{(0)} \cap H_+) \cup \partial(D_{-1}^{(0)} \cap H_-). \tag{3.68}$$

In (3.68) the interval $[-iy_1, iy_1]$ is covered twice. We have

$$\partial(D_1^{(0)} \cap H_+) = \Gamma_1 \cup \Gamma_{01} \cup \Gamma_{04} \cup [-iy_1, y_1] \tag{3.69}$$

with $\Gamma_1 := \Gamma_{11}$ and Γ_{01}, Γ_{04} the two subarcs of Γ_{12} that connect the two branch points z_1 and z_4 with iy_1 and $-iy_1$, respectively. The arcs Γ_{1k}, $k = 1, 2, 3$, have been introduced in Lemma 3.5. Symmetrically to (3.69) we have

$$\partial(D_{-1}^{(0)} \cap H_-) = \Gamma_{-1} \cup \Gamma_{02} \cup \Gamma_{03} \cup [-iy_1, y_1] \tag{3.70}$$

with $\Gamma_{-1} := \Gamma_{-1,1}$ and Γ_{02}, Γ_{03} the two subarcs of $\Gamma_{-1,2}$ that connect the two branch points z_2 and z_3 with iy_1 and $-iy_1$, respectively. If we define

$$K_0 := \Gamma_{01} \cup \cdots \cup \Gamma_{04} \cup [-iy_1, iy_1], \tag{3.71}$$

then the assertions (i)–(iv) of Lemma 2.6 are proved.

The numerical value (2.22) for y_1 has been calculated by a procedure based on Theorem 2.10.

The harmonicity of the function $h_{\max}$ in $\mathbb{C} \backslash \Gamma$ immediately follows from (3.66) and the properties of the branch functions $\hat{h}_j, j = -1, 0, 1$, established in Lemma 3.3. From there it also follows that if one crosses one of the subarcs of Γ, then the harmonic continuation of $h_{\max}$ is always different from $h_{\max}$ on the other side of the subarc. Therefore $h_{\max}$ cannot be harmonic in any neighborhood of points $z \in \Gamma$.

It has already been mentioned before Lemma 2.6 that representation (2.19) of $h_{\max}$ follows from the general theory of potential theory. As a maximum of harmonic functions, the function $h_{\max}$ has to be subharmonic (cf. [18, Chapter 2]), and representation (2.19) itself is an immediate consequence of the Poisson–Jensen formula (cf. [18, Theorem 4.5.1]). $\square$

3.4. *Proof of Theorem 2.8*

We shall use the same notations as in the proof of Lemma 2.6. The functions $h_j, j = -1, 0, 1, \infty$, which have been introduced in Definition 2.7, will be represented by pieces of the functions $\hat{h}_j, j = -1, 0, 1$, introduced in Lemma 3.3. The verification of representations (2.31) and (2.32) will take a central place in the proof.

We start with the investigation of the function h_1. By definition, this function is harmonic in $\mathbb{C} \backslash \Gamma_1$. The arc Γ_1 separates the two domains $D_1^{(0)}$ and $D_1^{(1)}$. We show that

$$h_1(z) = \begin{cases} \hat{h}_0(z) & \text{for } z \in D_1^{(1)}, \\ \hat{h}_1(z) & \text{for } z \in \mathbb{C} \backslash D_1^{(1)}. \end{cases} \tag{3.72}$$

Indeed, let $\tilde{h}_1$ denote the right-hand side of (3.72). From the introduction of the functions $\hat{h}_0$ and $\hat{h}_1$ in Lemma 3.3 we know that $\hat{h}_0$ is the harmonic continuation of $\hat{h}_1$ across the open arc $(\Gamma_{\infty 1} \cup \Gamma_{\infty 4}) \setminus \{z_1, z_4\}$ and vice versa. Therefore $\tilde{h}_1$ is harmonic on $(\Gamma_{\infty 1} \cup \Gamma_{\infty 4}) \setminus \{z_1, z_4\}$. From development (2.16) in Lemma 2.5 together with the second estimate of (3.24) in Lemma 3.3 we conclude that $\tilde{h}_1 = \hat{h}_0$ in $D_1^{(1)}$ in a neighborhood of infinity. Hence, at infinity the function $\tilde{h}_1$ has the development

$$\tilde{h}_1(z) = 3\,\mathrm{Re}(z) + \log|z| + O\left(\frac{1}{|z|}\right) \quad \text{as } |z| \to \infty, \tag{3.73}$$

which shows that the difference of the functions $\tilde{h}_1$ and h_1 of Definition 2.7 is harmonic in a neighborhood of infinity. Since $\tilde{h}_1$ has an harmonic extension to the whole domain $\mathbb{C} \setminus \Gamma_1$, it follows that (3.72) holds true.

Comparing (3.72) with (3.64) shows that $h_{\max} = h_1$ in $D_1^{(1)}$ and $(D_1^{(0)} \cap H_+)$. Thus, the two functions h_1 and $h_{\max}$ are identical in a neighborhood of the open arc $\Gamma_1 \setminus \{z_1, z_4\}$. From this and representation (2.19), together with (2.28), it then follows that

$$h_1(z) = \widetilde{h_{01}}(z) + \int \log|z - x|\, d\nu_1(x), \tag{3.74}$$

where $\widetilde{h_{01}}$ is a function harmonic in $\mathbb{C}$. A comparison of (3.74) with (3.73) then further implies that

$$h_1(z) = 3\,\mathrm{Re}(z) + \int \log|z - x|\, d\nu_1(x) \tag{3.75}$$

and

$$||\nu_1|| = 1, \tag{3.76}$$

which proves representation (2.31) for $j = 1$, and further shows that ν_1 is a probability measure.

In an identical way representation (2.31) can be proved for $j = -1$, and in the same way it is shown that ν_{-1} is a probability measure. In this part of the proof we have to verify that

$$h_{-1}(z) = \begin{cases} \hat{h}_0(z) & \text{for } z \in D_{-1}^{(-1)}, \\ \hat{h}_1(z) & \text{for } z \in \mathbb{C} \setminus D_{-1}^{(-1)} \end{cases} \tag{3.77}$$

instead of (3.72).

The function h_0 is harmonic in $\mathbb{C} \setminus K_0$. The set K_0 separates the four domains $D_1^{(0)} \cap H_+$, $D_{-1}^{(0)} \cap H_-$, $D_1^{(-1)} \cap D_{-1}^{(1)} \cap \{\mathrm{Im}(z) > 0\}$, and $D_1^{(-1)} \cap D_{-1}^{(1)} \cap \{\mathrm{Im}(z) < 0\}$. We shall show that

$$h_0(z) = \begin{cases} \hat{h}_1(z) & \text{for } z \in D_1^{(1)} \cup (D_1^{(0)} \cap H_+), \\ \hat{h}_0(z) & \text{for } z \in D_1^{(-1)} \cap D_{-1}^{(1)}, \\ \hat{h}_{-1}(z) & \text{for } z \in D_{-1}^{(-1)} \cup (D_{-1}^{(0)} \cap H_-). \end{cases} \tag{3.78}$$

Indeed, let $\tilde{h}_0$ denote the right-hand side of (3.78). It then follows again from the introduction of the three functions $\hat{h}_j, j = -1, 0, 1$, in Lemma 3.3 that the function $\tilde{h}_0$ can be harmonically continued across the open arcs $\Gamma_{\infty l} \backslash \{z_l, \infty\}$, $l = 1, \ldots, 4$. Since Γ_1 separates $D_1^{(1)}$ from $D_1^{(0)}$ and Γ_{-1} separates $D_{-1}^{(-1)}$ from $D_{-1}^{(0)}$, it follows again from the introduction of the three functions $\hat{h}_j, j = -1, 0, 1$, in Lemma 3.3 that $\tilde{h}_0$ can also be harmonically continued across the two open arcs $\Gamma_1 \backslash \{z_1, z_4\}$ and $\Gamma_{-1} \backslash \{z_2, z_3\}$. With this observation and definition (3.71) we see that the function $\tilde{h}_0$ has an harmonic extension to the whole domain $\mathbb{C} \backslash K_0$.

From development (2.15) in Lemma 2.5 and from (3.51) in Lemma 3.5 we conclude that $\tilde{h}_0 = \hat{h}_0$ in $D_1^{(-1)} \cap D_{-1}^{(1)}$ in a neighborhood of infinity. Hence, the function $\tilde{h}_0$ has the development

$$\tilde{h}_0(z) = \log 2 + \log |z| + O\left(\frac{1}{|z|}\right) \quad \text{as } |z| \to \infty, \tag{3.79}$$

which shows that in a neighborhood of infinity the function $\tilde{h}_0$ is identical with the function h_0 of Definition 2.7. Since $\tilde{h}_0$ has an harmonic extension to $\mathbb{C} \backslash K_0$, it follows that (3.78) holds true.

From (2.28) and representation (2.19) it then follows that

$$h_0(z) = \widetilde{h_{00}}(z) + \int \log |z - x| \, dv_0(x), \tag{3.80}$$

where $\widetilde{h_{00}}$ is a function harmonic in $\mathbb{C}$. A comparison of (3.80) with (3.79) implies that

$$h_0(z) = \log 2 + \int \log |z - x| \, dv_0(x) \tag{3.81}$$

and

$$|v_0|| = 1, \tag{3.82}$$

which proves representation (2.31) for $j = 0$ and further that v_0 is a probability measure.

From Lemma 3.3 and (3.67) we know that $\hat{h}_0$ is harmonic in $\mathbb{C} \backslash (K_\infty \cup \{0\})$. From (2.17) in Lemma 2.5 and from the introduction of h_∞ in Definition 2.7 we derive that $\hat{h}_0$ is identical with h_∞ in a neighborhood of the origin $z = 0$. Since the function $\hat{h}_0$ is harmonic throughout $D_\infty \backslash \{0\} = \mathbb{C} \backslash (K_\infty \cup \{0\})$, it follows that

$$h_\infty(z) = \hat{h}_0(z) \quad \text{for } z \in \mathbb{C} \backslash K_\infty. \tag{3.83}$$

From (2.17) in Lemma 2.5 we further learn that at the origin the function h_∞ has a logarithmic singularity with residue 3. From (3.50) and (3.83) we conclude that

$$h_{\max}(z) = h_\infty(z) \quad \text{for } z \in \mathbb{C} \backslash (K_\infty \cup \overline{D_1^{(0)} \cup D_{-1}^{(0)}}). \tag{3.84}$$

As a consequence, it follows from (2.29) and representation (2.19) that all singularities of the function h_∞, except the one at the origin, are represented by the positive measure v_∞. With this knowledge we see that an application of the

Poisson–Jensen Formula to the function h_∞ yields representation (2.32). We note that $-\log|R(z-t)/(R^2-\bar{t}z)|$ is the Green function $g(z,t)$ in the disc $\{|z|<R\}$.

From the second estimate in (3.24) in Lemma 3.3, representation (2.32), and definition (2.20) of the function h_R, we deduce that

$$\lim_{R\to\infty}\left|h_R(z)-\log R-\frac{1}{2\pi R}\int_{|t|=R}\max(-3\,\mathrm{Re}(t),\log(2),3\,\mathrm{Re}(t))\,\frac{R^2-|z|^2}{|z-t|^2}\,d|t|\right| = 0 \tag{3.85}$$

locally uniformly for $z\in\mathbb{C}$. Indeed, from the second estimate of (3.24) and (3.83) we know that

$$h_\infty(z)=\log R+\max(-3\,\mathrm{Re}(t),\log(2),3\,\mathrm{Re}(t))+O\left(\frac{1}{R}\right) \quad \text{as } R\to 0 \text{ and } |z|=R, \tag{3.86}$$

which together with the properties of the Poisson kernel and (2.20) proves (3.85).

Limit (3.85) together with representation (2.32) and definition (2.4) of the function $h(v_\infty;\cdot)$ shows that $h_\infty=h(v_\infty;\cdot)$, which proves assertion (iii) in Theorem 2.8.

Next, we derive representations (2.30) of the density functions dv_j/ds of the four measures $v_j, j=-1,0,1,\infty$. In order to have a more concrete notation we first consider the case $j=1$. The function h_1 has harmonic continuations from both sides of the open arc $\Gamma_1\backslash\{z_1,z_4\}$. By h_{1+} and h_{1-} we denote the boundary values of h_1 from both sides of $\Gamma_1\backslash\{z_1,z_4\}$, and by $\partial/\partial n_+$ and $\partial/\partial n_-$ the corresponding normal derivatives. From the Green formula and representation (2.31) it follows that the measure v_1 is given by

$$dv_1(z)=\frac{1}{2\pi}\left|\frac{\partial}{\partial n_+}h_{1+}(z)+\frac{\partial}{\partial n_-}h_{1-}(z)\right|ds_z, \quad z\in\Gamma_1, \tag{3.87}$$

where ds is the line element on Γ_1. Indeed, let $G\subseteq D_1$ be a domain with smooth boundary, and let $\partial/\partial n$ denote the normal derivatives to ∂G. Set $g_1(t):=h_1(t)-3\,\mathrm{Re}(t)$, $g_2(t):=\log|z-t|, t\in G$. From the Green formula we then know that

$$\oint_{\partial G}\left[\frac{\partial g_1}{\partial n}g_2-g_1\frac{\partial g_2}{\partial n}\right](t)\,ds_t=0, \tag{3.88}$$

which yields

$$g_1(z)=c_1+\frac{1}{2\pi}\oint_{\partial G}\frac{\partial g_1}{\partial n}(t)\log|z-t|\,ds_t \tag{3.89}$$

with $c_1:=(g_2-g_1)(\infty)$. If the domain $G\subseteq D_1$ is extended so that it exhausts the domain D_1, then a comparison of (3.89) with representation (2.31) yields (3.87).

In an analogous way we can derive representations for the density functions of the three measures v_0, v_{-1}, and v_∞. In the cases of the measures v_0 and v_∞, the analogues to formula (3.87) do not operate on a single arc, but on all subarcs of the sets K_0 and K_∞ as stated in (3.71) and (3.67). The technical details should cause no difficulties.

From representation (2.32) together with the behavior of the function h_∞ in a neighborhood of infinity, as has been shown in the second estimate of (3.24) in

Lemma 3.3, we conclude that the measure ν_∞ has to have infinite mass, which proves (2.29).

In assertion (i) of Theorem 2.8 it is stated that the four measures $\nu_j, j = -1, 0, 1, \infty$, which have been defined in (2.28) with the help of representation (2.19) of the function $h_{\max}$, are identical with the measures that appear in the Theorems 2.1 and 2.2. This fact can only be proved in Section 4, when the proofs of the Theorems 2.1, 2.2, and 2.8 have been completed. The proof then follows in a simple and natural way. We note that in the proofs of Section 4 it is assumed that the four measures $\nu_j, j = -1, 0, 1, \infty$, are defined by (2.28) and representation (2.19), which is an analytic definition independent of the asymptotics considered in the Theorems 2.1 and 2.2. □

3.5. *Proof of Theorem 2.10*

The tangential directions (2.37) have already been proved in (3.49) of Lemma 3.5. Thus, it remains only to prove formulae (2.38)–(2.42).

Form Lemma 2.6 we know that $\Gamma \backslash \{z_1, \ldots, z_4, iy_1, -iy_1, \infty\}$ consists of open, analytic Jordan arcs. In order to have concrete notations, we first concentrate on the subarc $\Gamma_1 \subseteq \Gamma$. Let $\partial/\partial t$ denote the tangential derivatives along $\Gamma_1 \backslash \{z_1, z_4\}, t = t_z \in \partial\mathbb{D}$ the tangent vector at $z \in \Gamma_1 \backslash \{z_1, z_4\}$, and h_1 the branch function of the function $h \circ \pi^{-1}$ introduced in Definition 2.7, which is harmonic in the domain $D_1 = \bar{\mathbb{C}} \backslash \Gamma_1$. As in the proof of Theorem 2.8 by h_{1+} and h_{1-} we denote the boundary values of h_1 from both sides of Γ_1.

From Definition 2.7 and Lemma 2.6 we know that the open arc $\Gamma_1 \backslash \{z_1, z_4\}$ is contained in the domain D_0 and also in the domain D_∞. We deduce from (3.72) and (3.83) on one side, and from (3.72) and (3.78) on the other side that

$$h_{1+}(z) = h_\infty(z) \quad \text{and} \quad h_{1-}(z) = h_0(z) \quad \text{for all } z \in \Gamma_1 \backslash \{z_1, z_4\}. \tag{3.90}$$

Since $h_{1+}(z) = h_{1-}(z)$ for $z \in \Gamma_1$, formula (2.38) then follows directly from (3.90) and assertion (ii) of Lemma 3.2 for all $z \in \Gamma_1 \backslash \{z_1, z_4\}$.

For the other subarcs of $\Gamma \backslash \{z_1, \ldots, z_4, iy_1, -iy_1, \infty\}$ the corresponding formulae in (2.39)–(2.42) can be derived in exactly the same way, only that the identities in (3.90) have to be modified in an appropriate way. □

3.6. *Proof of Theorem 2.11*

Like formulae (2.38)–(2.42) of Theorem 2.10 followed from assertion (ii) of Lemma 3.2, so formulae (2.44)–(2.47) of Theorem 2.11 analogously follow from assertion (iii) of Lemma 3.2 together with representations (2.30) in Theorem 2.8. □

4. Proofs, Part II

In the present and last section we prove the key result of the paper, which is Theorem 2.9. Theorems 2.1 and 2.2 belong to the same circle of problems, but they are practically corollaries of Theorem 2.9, and their proofs will be rather short.

4.1. The saddle point method

The main tool in the proof of Theorem 2.9 is the saddle point method, which we formulate in the next theorem in a slightly specialized version, which is adapted to the needs of our analysis.

Theorem 4.1 (Saddle Point Theorem). *Let the integral I_n be of the form*

$$I_n = \int_\Gamma g(v)G(v)^n \, dv, \tag{4.1}$$

and assume that its elements, i.e., the two functions g, G, and the integration path Γ depend on a parameter z. We make the following specific assumptions:

(i) *The parameter z is assumed to vary in an open set $V \subseteq \mathbb{C}$.*

(ii) *There exists a family $\Gamma = \Gamma_z, z \in V$, of Jordan arcs or Jordan curves. For each $z \in V$ a point $\zeta_0 = \zeta_{0,z} \in \Gamma_z$ is singled out. If Γ_z is a Jordan arc, then $\zeta_{0,z}$ is assumed to be an inner point of Γ_z. The arcs or curves Γ_z and the points $\zeta_{0,z}$ are assumed to depend continuously on z for $z \in V$.*

(iii) *For each $z \in V$ the functions $g = g(\cdot, z)$ and $G = G(\cdot, z)$ are assumed to be analytic in an open neighborhood U of Γ_z. We assume that $g(\zeta_0) = g(\zeta_{0,z}, z) \neq 0$ and that*

$$|G(\zeta_{0,z}, z)| > |G(\zeta, z)| \quad \text{for all } \zeta \in \Gamma_z \backslash \{\zeta_{0,z}\}. \tag{4.2}$$

The functions g and G are assumed to depend continuously on $z \in V$, but the neighborhood U is assumed to be fixed, i.e., we consider the same neighborhood U for all $z \in V$.

(iv) *For each $z \in V$ the function $G(\cdot, z)$ is assumed to have a non-degenerated critical point at $\zeta_{0,z}$, i.e., $G'(\zeta_0) = \partial G(\zeta_{0,z}, z)/\partial\zeta = 0$ and $G''(\zeta_0) = \partial^2 G(\zeta_{0,z}, z)/\partial\zeta^2 \neq 0$.*

Under these assumptions we have the estimate

$$I_n = \sqrt{\frac{-2\pi G(\zeta_0)}{nG''(\zeta_0)}} g(\zeta_0) G(\zeta_0)^n \left(1 + O\left(\frac{1}{n}\right)\right) \quad \text{as } n \to \infty \tag{4.3}$$

for integral (4.1) *with the sign of the square root chosen in such a way that*

$$\left| \arg \sqrt{\frac{-2\pi G(\zeta_0)}{nG''(\zeta_0)}} - \arg(dt_{\zeta_0}) \right| < \frac{\pi}{4}. \tag{4.4}$$

In (4.4) *dt_{ζ_0} denotes the tangential line element on Γ at the point ζ_0 and it has to have the same orientation as used in integral* (4.1). (*Because of assumption* (4.2), *such a choice of the sign is always possible.*) *By $O(\cdot)$ we denote Landau's symbol 'big oh', and* *in* (4.3) *the symbol holds uniformly for parameter values z varying on compact subsets of V.*

A discussion of the saddle point method can be found in [17] or [25, Chapter 2]. The theorem stated here is special in so far, as we consider only non-degenerated critical points (i.e., proper saddle points) and the precision of the asymptotic estimate (4.3) takes into account only the first term in a possible development in powers of $1/n$. In the proofs given in [17,25], it can be verified rather easily that the continuous dependence on $z \in V$ of all elements in (4.1) implies the uniformity of the Landau symbol in (4.3).

For the application of the saddle point method in the proof of Theorem 2.9 we need several preliminary transformations and definitions. Among them an appropriate definition of the integration paths $C_j, j = -1, 0, 1, \infty$, in integrals (1.9)–(1.12) is of great importance.

4.2. Transformation of the integral representations

Representations (1.9)–(1.12) suggest that we have to study the asymptotic behavior of the integral

$$\widetilde{I}_n = \frac{n! 2^{n+1}}{2\pi i 3^n n^n} \oint_C \frac{e^{3nzv}\, dv}{v^{n+1}(v^2-1)^{n+1}} \quad \text{for } n \to \infty \tag{4.5}$$

with C one of the four integration paths $C_j, j = -1, 0, 1, \infty$, introduced in (1.9)–(1.12). We can assume that the four integration paths C_j are Jordan curves, the first three are positively, and the last one is negatively oriented. We have $\mathrm{Int}(C_j) \cap \{-1, 0, 1\} = \{j\}$ for $j = -1, 0, 1$, and $\mathrm{Int}(C_\infty) \supseteq \{-1, 0, 1\}$.

Cauchy's Theorem guarantees large freedom for the choice of the curves $C_j, j = -1, 0, 1, \infty$. In order to apply the saddle point method successfully, the curves have to be chosen in a rather specific way. This problem will be addressed in the next Section 4.3. Here, we rewrite integral (4.5) in a way that allows connections with the definitions of the Riemann surface $\mathcal{R}$ and the functions ψ and h that have been introduced in Definitions 2.3 and 2.4 and have been studied intensively in Section 3.

By using Stirling's formula

$$n! = n^n e^{-n} \sqrt{2\pi n}\left(1 + O\left(\frac{1}{n}\right)\right) \quad \text{as } n \to \infty \tag{4.6}$$

the quotient $n!/n^n$ in (4.5) can be rewritten, and we arrive at the integral

$$I_n := -i\sqrt{\frac{2n}{\pi}} \oint_C \left[\frac{e^{3zv-1+\log(2/3)}}{v(v^2-1)}\right]^n \frac{dv}{v(v^2-1)}, \tag{4.7}$$

which because of (4.6) satisfies the relation

$$\widetilde{I}_n = I_n\left(1 + O\left(\frac{1}{n}\right)\right) \quad \text{as } n \to \infty. \tag{4.8}$$

In (4.7) C denotes one of the four integration paths $C_j, j = -1, 0, 1, \infty$, that have already been used in integral (4.5). We will analyze integral (4.7) somewhat further. A comparison of (4.7) with (4.1) leads to the following definitions:

$$G(v) = G(v, z) := \frac{e^{3zv-1+\log(2/3)}}{v(v^2-1)} = e^{a_z(v)} \tag{4.9}$$

with

$$a_z(v) := 3zv - 1 + \log\frac{2}{3v(v^2-1)} \tag{4.10}$$

and

$$g(v) := \frac{-i\sqrt{2n}}{\sqrt{\pi}v(v^2-1)}. \tag{4.11}$$

The function G depends on z as external parameter, while the function g is independent of z. In most of the analysis that follows, the parameter $z \in \mathbb{C}$ can be considered as fixed.

A point $v = v_z \in \mathbb{C}$ is a *critical point* of G if

$$G'(v) = G(v)a_z'(v) = G(v)\left(3z - \frac{3v^2-1}{v(v^2-1)}\right) = 0, \tag{4.12}$$

or equivalently if $zv(v^2-1) - v^2 + 1/3 = 0$, which is exactly Eq. (2.11). Via (2.5) this equation is linked with the definitions of the Riemann surface $\mathscr{R}$ and the mapping $\psi : \mathscr{R} \to \bar{\mathbb{C}}$ in Definition 2.3.

For $z \in \bar{\mathbb{C}} \backslash \{z_1, \ldots, z_4\}$ equation (2.11) has three different solutions, which for the moment will be denoted by $v_z^{(l)}, l = 1, 2, 3$. (Below, we shall introduce a different numeration.) The solutions $v_z^{(l)}$ depend on z, and from (2.11), (2.5), and (2.6) we know that the $v_z^{(l)}$ are preimages of the point z under the function $\pi \circ \psi^{-1}$, i.e., function (2.5). Indeed, from the definition of the function ψ in (2.6) it follows that

$$\{v_z^{(1)}, v_z^{(2)}, v_z^{(3)}\} = \psi \circ \pi^{-1}(\{z\}). \tag{4.13}$$

If $z \in \{z_1, \ldots, z_4\}$, then two of the three elements in the set on the left-hand side of (4.13) coincide. For $z \notin \{z_1, \ldots, z_4\}$ all three critical points $v_z^{(l)}, l = 1, 2, 3$, are simple solutions of Eq. (4.12), and therefore it follows that $G''(v_z^{(l)}) \neq 0$ for $l = 1, 2, 3$. In case of $z \in \{z_1, \ldots, z_4\}$, the set contains one simple and one double solution of Eq. (4.12). Let $v_z^{(1)}$ denote the simple and $v_z^{(2)}$ the double solution, then we have $G''(v_z^{(1)}) \neq 0$ and $G''(v_z^{(2)}) = 0$. As consequence the following lemma holds true.

Lemma 4.2. *If $z \notin \{z_1, \ldots, z_4\}$, then all three critical points $v_z^{(l)}, l = 1, 2, 3$, are non-degenerated. If $z \in \{z_1, \ldots, z_4\}$, then there exist only two different critical points $v_z^{(l)}$, $l = 1, 2$. One is degenerated and the other one is non-degenerated.*

Next, we consider the value of the function $\operatorname{Re} a_z(v)$ at critical points $v = v_z^{(l)}, l = 1, 2, 3$. We deduce from (4.10) and (4.12) that

$$\begin{aligned} a_z(v_z^{(l)}) &= \frac{v_z^{(l)}(3(v_z^{(l)})^2 - 1)}{v_z^{(l)}((v_z^{(l)})^2 - 1)} - 1 + \log \frac{2}{3v_z^{(l)}((v_z^{(l)})^2 - 1)} \\ &= \frac{2(v_z^{(l)})^2}{(v_z^{(l)})^2 - 1} + \log \frac{2}{3v_z^{(l)}((v_z^{(l)})^2 - 1)} \\ &= u(v_z^{(l)}), \quad l = 1, 2, 3, \end{aligned} \tag{4.14}$$

where u is the function introduced in (2.13) of Definition 2.4. Thus, it follows from the definition of the function $h : \mathscr{R} \to \bar{\mathbb{R}}$ in (2.12) that

$$\operatorname{Re} a_z(v_z^{(l)}) = \operatorname{Re} u(v_z^{(l)}) = h(\zeta), \quad l = 1, 2, 3, \tag{4.15}$$

with $\zeta \in \mathscr{R}$ chosen from the three points of $\pi^{-1}(\{z\})$ in such a way that $\psi(\zeta) = v_z^{(l)}$. From (4.15) we learn that the function h introduced in Definition 2.4 is equal to $\log |G(v)|$ if $v = v_z^{(l)}$ is a critical point. The values, which are assumed by G at the critical points, have great significance in the saddle point method. For the use in this method it will turn out to be preferable to choose a numeration of the critical points $v_z^{(l)}, l = 1, 2, 3$, that corresponds to the definition of the branch functions ψ_j and $h_j, j = -1, 0, 1, \infty$, of the multi-valued functions $\psi \circ \pi^{-1}$ and $h \circ \pi^{-1}$, respectively, that have been introduced in Definition 2.7. These branch functions correspond to the four domains $D_j, j = -1, 0, 1, \infty$, that have also been introduced in Definition 2.7.

Definition 4.3. Let $\psi_j, j = -1, 0, 1, \infty$, be the branch functions introduced in (2.25) of Definition 2.7. Let further $z \in D_j, j = -1, 0, 1, \infty$, then from the three critical points in (4.13) we denote that point $v_z^{(l)}$ as $v_j = v_{j,z}$ that is equal to $\psi_j(z)$, i.e., we define

$$v_{j,z} \coloneqq \psi_j(z), \quad j = -1, 0, 1, \infty. \tag{4.16}$$

Remark. We learn from (4.13) that all critical points of G for a given z are preimages of the multi-valued function $\pi \circ \psi^{-1}$, and therefore also preimages of the branch functions ψ_j. If $z \in \bar{\mathbb{C}} \setminus \{z_1, \dots, z_4\}$, then exactly three of points (4.16) are different. If $z \notin K_{-1} \cup \dots \cup K_\infty$, then the point z is covered by all four domains $D_j, j = -1, 0, 1, \infty$, and formally we have four critical points $v_{j,z}$ in (4.16). Of these four points only three are different. We note that from Remark 2 to Definition 2.7 we know that $\{z_1, \dots, z_4\} \cap D_j = \emptyset$ for all $j = -1, 0, 1, \infty$.

Identity (4.15) is reformulated in the following lemma with the new numeration of critical points that has been introduced in (4.16).

Lemma 4.4. *With each critical point* $v_{j,z}, j = -1, 0, 1, \infty$, *is associated the value* $c_{j,z} := \operatorname{Re} a_z(v_{j,z})$ *as the critical level at* $v_{j,z}$. *We have*

$$\operatorname{Re} a_z(v_{j,z}) = h_j(z) \quad \text{for } z \in D_j, \ j = -1, 0, 1, \infty, \tag{4.17}$$

with h_j *the functions introduced in Definition* 2.7.

4.3. Definition of the integration paths C_j

In the Lemmas 4.5 and 4.6 we shall prove the existence of integration paths $C_j, j = -1, 0, 1, \infty$, for integral (4.7) with properties that are appropriate for the saddle point method.

Pieces of the integration paths will be level lines of the harmonic function $\operatorname{Re} a_z(\cdot)$ introduced in (4.10). For $c \in \mathbb{R}$ and $z \in \mathbb{C}$, we denote the system of curves

$$L_c = L_{c,z} := \{v \in \bar{\mathbb{C}} \mid \operatorname{Re} a_z(v) = c\} \tag{4.18}$$

as *level lines*, and the sets

$$M_c = M_{c,z} := \{v \in \bar{\mathbb{C}} \mid \operatorname{Re} a_z(v) \geqslant c\} \tag{4.19}$$

as the *filled in level lines* of the harmonic function $\operatorname{Re} a_z(\cdot)$. The curves L_c are different from those introduced in (3.30) and were used only localy in the proof of Lemma 3.3. At this earlier place the function $\operatorname{Re} u$ has been used at the pace of $\operatorname{Re} a_z(v)$ in (4.18) and (4.19). Both definitions should not be mixed up.

Lemma 4.5. *For* $z \in \mathbb{C} \backslash \{0\}$ *and* $c \in \mathbb{R}$ *sufficiently large, the following three assertions hold true*:

(i) *Each of the two sets* L_c *and* M_c *consists of four components* $L_c^{(j)}$ *and* $M_c^{(j)}, j = -1, 0, 1, \infty$, *respectively. The numeration can be chosen in such a way that*

$$j \in M_c^{(j)} \quad \text{and} \quad L_c^{(j)} = \partial M_c^{(j)} \quad \text{for } j = -1, 0, 1, \infty. \tag{4.20}$$

(ii) *Each of the three components* $L_c^{(j)}, j = -1, 0, 1$, *is an analytic Jordan curve, approximately equal to a small ellipsis surrounding the point* $v = j$. *With a positive orientation the curve* $L_c^{(j)}$ *can be taken as integration path* C_j *in integral* (4.7).

(iii) *Let* $R > 1$ *be chosen so that all critical points* (4.16) *are contained in* $\{|z| < R\}$. *Then the contour* $L_c^{(\infty,R)} := \partial(\{|v| \geqslant R\} \cup M_c^{(\infty)})$ *is a piece-wise analytic Jordan curve having* ∞ *in its exterior and the set* $\{-1, 0, 1\}$ *in its interior. With a negative orientation the curve* $L_c^{(\infty,R)}$ *can be taken as integration path* C_∞ *in integral* (4.7).

If $z = 0$ *and* $c \in \mathbb{R}$ *again sufficiently large, then the two sets* L_c *and* M_c *have only three components*; *the components* $L_c^{(\infty)}$ *and* $M_c^{(\infty)}$ *are missing. For the remaining three components* $L_c^{(j)}, j = -1, 0, 1$, *assertion* (ii) *and relation* (4.20) *holds true.*

Proof. The equation $\operatorname{Re} a_z(v) = c$ in (4.18) can be rewritten with the help of (4.10) as

$$\log\left|\frac{1}{v(v^2-1)}\right| = -3\operatorname{Re}(zv) + 1 + \log\frac{2}{3} + c. \tag{4.21}$$

The left-hand side of (4.21) is a rather simply constructed potential in the v-plane with logarithmic singularities at the three points $v = -1, 0, 1$. The right-hand side of (4.21) is a real linear function depending on the parameter $z \in \mathbb{C}$ and the constant $c \in \mathbb{R}$.

If $z \neq 0$, then assertion (i) follows immediately from the structure of both sides of (4.21). Also assertion (ii) follows directly from (4.21) since for z fixed and $c \in \mathbb{R}$ sufficiently large, it is not difficult to see that the components $L_c^{(j)}$ approximately are small ellipses around the points $v = j = -1, 0, 1$.

In order to verify assertion (iii) we introduce the new independent variable $\hat{v}$ defined by $\hat{v} := v/c$. It then follows from (4.21) that the sets $\hat{M}_c^{(\infty)} := \{\hat{v} \in \bar{\mathbb{C}} \mid c\hat{v} \in M_c^{(\infty)}\}$ converge to the set $\hat{M} := \{\hat{v} \in \bar{\mathbb{C}} \mid 3\operatorname{Re}(z\hat{v}) \geqslant 1\}$ if c tends to infinity. Assertion (iii) follows directly from the simple structure of the set $\hat{M}$.

If $z = 0$, then the right-hand side of (4.21) is constant, and the conclusions in the lemma are immediate. □

The curves $L_c^{(j)}$ defined in Lemma 4.5 are admissible as integration paths for integral (4.7), but they do not satisfy the requirements of the saddle point method since condition (4.2) is not satisfied. However, modifications are possible that result in curves with the required properties. The curve $L_c^{(j)}$ has to be pushed downwards by lowering $c \in \mathbb{R}$ until $L_c^{(j)}$ hits a critical point for the first time. At this stage a merger of some of the components of the set M_c takes place. If one pushes $L_c^{(j)}$ slightly further, then after some local modifications one arrives at a curve that has the desired properties. Details of this procedure will be given in the proof of the next lemma.

Lemma 4.6. *We choose* $j \in \{-1, 0, 1, \infty\}, z \in D_j$, *and in case of* $j = \infty$ *we assume that* $z \neq 0$. *Let* G *be the function introduced in* (4.9). *The following three assertions hold true*:

(i) *There exists a piece-wise analytic Jordan curve* $C_j = C_{j,z}$, *which is admissible as integration path in integral* (4.7).

(ii) *The critical point* $v_j = v_{j,z}$ *defined in* (4.16) *lies on* C_j, *and we have*

$$|G(v_j)| > |G(v)| \quad \textit{for all } z \in C_j \backslash \{v_j\}. \tag{4.22}$$

(iii) *Both, the curve* $C_{j,z}$ *and the critical point* $v_{j,z}$ *depend continuously on the parameter* z.

Proof. We start by defining and considering critical curves. Then we show in an auxiliary lemma, how the critical points are associated with the critical curves. At last we show how the critical curve can be modified so that inequality (4.22) holds true.

If $c \in \mathbb{R}$ is large, then we know from Lemma 4.5 that the curves $L_c^{(l)}, l = -1, 0, 1$, and also the curve $L_c^{(\infty,R)}$ for R sufficiently large, are small in the cordal metric, and we can assume that no critical points lie on these curves. The curves vary continuously with the parameter z, and each component $M_c^{(l)}, l = -1, 0, 1, \infty$, of M_c grows monotonically if c is moved downwards. At some value $\tilde{c}_j = \tilde{c}_{j,z} \in \mathbb{R}$, the curve $L_c^{(j)}$, or $L_c^{(\infty,R)}$ in case of $j = \infty$, hits a critical point for the first time. This critical point is denoted by $\tilde{v}_{j,z}$ and called critical point of first contact (for j), and further the value $\tilde{c}_j$ is called critical level. At the critical level $c = \tilde{c}_j$ the connectivity of the set M_c changes. We define

$$\tilde{M}_j = \tilde{M}_{j,z} := \bigcup_{c > \tilde{c}_j} M_c^{(j)} \quad \text{in case of } j \in \{-1, 0, 1\},$$

and

$$\tilde{M}_\infty = \tilde{M}_{\infty,z} := \{|z| > R\} \cup \bigcup_{c > \tilde{c}_j} M_c^{(\infty)} \quad \text{in case of } j = \infty. \tag{4.23}$$

Here, $R > 1$ is the same constant as introduced in part (iii) of Lemma 4.5. The *critical curve* $\tilde{L}_j$ is defined as

$$\tilde{L}_j = \tilde{L}_{j,z} := \partial \tilde{M}_j. \tag{4.24}$$

It is immediate that the curves $L_c^{(j)}$, or $L_c^{(\infty,R)}$ in case of $j = \infty$, converge to the critical curve $\tilde{L}_j$ if c tends to $\tilde{c}_j$ from above. The curve $\tilde{L}_j$ is piece-wise analytic, and it is a Jordan curve. The next lemma gives informations about the association between the critical points and the critical curve $\tilde{L}_j$.

Lemma 4.7. *Let $v_j = v_{j,z} = \psi_j(z), z \in D_j$, be the critical point introduced in* (4.9) *of Definition* 4.3, *and $c_j = c_{j,z} = \operatorname{Re} a_z(v_{j,z}) = h_j(z)$ the critical level introduced in Lemma* 4.4. *We have*

$$v_{j,z} = \tilde{v}_{j,z} \tag{4.25}$$

and

$$c_{j,z} = \tilde{c}_{j,z}. \tag{4.26}$$

The critical curve $\tilde{L}_j = \tilde{L}_{j,z}$ contains exactly one critical point, which is $v_{j,z}$, i.e., we have

$$v_{j,z} \in \tilde{L}_{j,z} \subseteq L_{\tilde{c}_{j,z}}, \quad \tilde{c}_{j,z} = h_j(z). \tag{4.27}$$

Lemma 4.7 will be proved below after we have finished the proof of Lemma 4.6.

In a neighborhood U_j of the critical point v_j we define the two Green lines $\gamma_{j\pm} = \gamma_{j\pm,z}$, $z \in D_j$, of the harmonic function $\operatorname{Re} a_z(\cdot)$ by the properties that

$$\operatorname{Im} a_z(v) = \operatorname{Im} a_z(v_j) \quad \text{for } v \in \gamma_{j\pm},$$

$$\operatorname{Re} a_z(v) \begin{cases} > \operatorname{Re} a_z(v_j) & \text{for } v \in \gamma_{j+} \setminus \{v_j\}, \\ < \operatorname{Re} a_z(v_j) & \text{for } v \in \gamma_{j-} \setminus \{v_j\}. \end{cases} \tag{4.28}$$

Since $\operatorname{Re} a_z(\cdot)$ is harmonic in U_j and has a non-degenerated critical point at v_j, it is immediate that the two Green lines $\gamma_{j\pm}$ are well defined. They are analytic Jordan arcs, and both arcs intersect orthogonally at the point v_j.

From Lemma 4.7 we know that there exists exactly one critical point $v_j \in \tilde{L}_j$. If $\epsilon > 0$ is sufficiently small, then the set $M_{c_j-\epsilon}$ contains exactly the same critical points as the set M_{c_j}, and the Green line γ_{j-} divides the set $M_{c_j-\epsilon}$ into at least two parts like it divides the neighborhood U_j of v_j. Note that from (4.28) we know that the Green line γ_{j-} is downward showing, i.e., the function $\operatorname{Re} a_z(\cdot)$ is monotonically decreasing on γ_{j-} as the argument moves away from v_j. By $\tilde{M}_{j,\epsilon}$ we denote the component of $\operatorname{Int}(\tilde{M}_{j,\epsilon} \backslash \gamma_{j-})$ that satisfies the relation $\tilde{M}_j \subseteq \tilde{M}_{j,\epsilon}$.

We next show that with an appropriate orientation the contour $\partial \tilde{M}_{j,\epsilon}$ can be taken as the integration path C_j in integral (4.7). Indeed, $\partial \tilde{M}_{j,\epsilon}$ is a piece-wise analytic Jordan curve, which consists of a piece of the level line $L_{c_j-\epsilon}$ and a piece of the Green line γ_{j-}. Hence, it is rather immediate from the construction of $\tilde{M}_j$ and $\tilde{M}_{j,\epsilon}$ that the contour $\partial \tilde{M}_{j,\epsilon}$ is an admissible integration path C_j. From (4.28) and the definition of the level line L_c in (4.18) we deduce that

$$
\begin{aligned}
&|G(v)| = e^{c_j-\epsilon} \quad \text{for } v \in L_{c_j-\epsilon} \cap \partial \tilde{M}_{j,\epsilon},\\
&e^{c_j-\epsilon} < |G(v)| < e^{c_j} \quad \text{for } v \in (\gamma_{j-} \cap \partial \tilde{M}_{j,\epsilon}) \backslash \{v_j\},\\
&|G(v_j)| = e^{c_j}.
\end{aligned} \tag{4.29}
$$

Inequality (4.22) follows from (4.29). The continuous dependence of $c_j = c_{j,z}$ and $v_j = v_{j,z}$ on the parameter z follows from the fact that the critical point $v_{j,z}$ is non-degenerated, and also from the analytic character of all elements in the definitions of the curve $C_j = C_{j,z}$ and the critical point $v_{j,z}$. $\square$

Proof of Lemma 4.7. For the given $j \in \{-1, 0, 1, \infty\}$ and $z \in D_j$, we first prove the two relations (4.25) and (4.26), and then we show that the critical curve $\tilde{L}_j$ contains only one critical point, which is $v_{j,z}$.

From the definition of the domain D_j in Lemma 3.3, it follows that $z \in D_j$ implies $z \notin \{z_1, \ldots, z_4\}$. Therefore we know that among the four critical points $v_{l,z}, l = -1, 0, 1, \infty$, introduced in (4.16) of Definition 4.3 there are exactly three pair-wise distinct ones, and each critical point is non-degenerated. From Definition 4.3 and Lemma 4.5 it follows immediately that the critical points $v_{l,z}, l = -1, 0, 1, \infty$, depend analytically, and the critical levels $c_{l,z}, l = -1, 0, 1, \infty$, depend harmonically on $z \in D_l$. As a consequence we know that also the critical point of first contact $\tilde{v}_{j,z}$ and the corresponding critical level $\tilde{c}_{j,z}$ introduced in the proof of Lemma 4.6 depend analytically and harmonically on z as long as there is no exchange of critical points of first contact $\tilde{v}_{j,z}$ at some point z. Such a change, however, is only possible if at the point z, where it takes place, the critical curve $\tilde{L}_j$ contains at least two critical points.

First, we consider the situation that $z \in D_j$ lies near infinity and $j \neq \infty$. From (4.16) and the properties of the functions $\psi_l(z), l = -1, 0, 1, \infty$, established in

Definition 2.7, it follows that for z sufficiently close to infinity the critical point $v_{l,z}$ lies near the point $v = l$, $l = -1, 0, 1$. Since the curve $L_c^{(j)}$ is a small quasi-circle around $v = j$ for $c \geqslant \tilde{c}_{j,z}$, it follows that $v_{j,z}$ is the critical point of first contact $\tilde{v}_{j,z}$. This proves the two identities (4.25) and (4.26) for z near ∞ and $j \neq \infty$.

If $z \in D_j$ lies near infinity and $j = \infty$, then among the three critical points $v_{l,z}, l = -1, 0, 1$, the critical point of first contact $\tilde{v}_{\infty,z}$ is the point with the greatest critical level $c_{l,z}$. In the proof of Theorem 2.8, and there especially in (3.84) and the identities leading to (3.84), it has been shown that we have $h_\infty = h_{\max} = (h_{-1}, h_0, h_1)$ in a neighborhood of infinity, and that among the three values $h_l(z), l = -1, 0, 1$, there is a unique maximum near infinity for all $z \in D_\infty$. With (4.16) it therefore follows that $v_{\infty,z} = h_\infty(z)$ is the critical point of first contact $\tilde{v}_{\infty,z}$, which proves (4.25) and (4.26) for z near ∞ and $j = \infty$.

Next, we consider the situation that $z \in D_j$ lies near to the origin, but $z \neq 0$. The function $\operatorname{Re} a_z(\cdot)$ then has the two critical points $v_{-1,z}$ and $v_{1,z}$ near the two points $-\sqrt{1/3}$ and $\sqrt{1/3}$, respectively, and a third critical point $v_{\infty,z}$ near infinity.

It follows from the simple structure of the function $\operatorname{Re} a_z(\cdot)$ that if $j \neq \infty$, then the critical point of first contact $\tilde{v}_{j,z}$ has to be one of the two points $v_{-1,z}$ and $v_{1,z}$. The critical level $\tilde{c}_{j,z}$ is bounded away from $-\infty$ for all z in a neighborhood of the origin. From (3.72), (3.77), and (3.78) in the proof of Theorem 2.8 we know that $h_0(z) = h_{-1}(z)$ for z close to the origin and $\operatorname{Re}(z) \leqslant 0$, and $h_0(z) = h_1(z)$ for z close to the origin and $\operatorname{Re}(z) \geqslant 0$.

In case of z near 0 and $j = \infty$, the critical point of first contact $\tilde{v}_{\infty,z}$ is identical with the critical point $v_{\infty,z}$ that is lying close to infinity. From (2.32) in Theorem 2.8 it follows that in this case the critical level $\tilde{c}_{\infty,z} = h_\infty(v_{\infty,z})$ tends to $-\infty$ if z tends to 0.

From the observations in the last two paragraphs we conclude that identities (4.25) and (4.26) hold true for z close to the origin. Further, we have shown that only in case of $j = \infty$ the values $\tilde{c}_{j,z} = h_j(v_{j,z})$ tend to $-\infty$ if z tends to 0.

It remains to show that the two identities (4.25) and (4.26) hold for all $z \in D_j$. From earlier conclusions we know that as long as there is no change from one critical point of first contact to another one at a certain z, the critical level $\tilde{c}_{j,z}$ is a harmonic function of $z \in D_j$, and the critical point of first contact $\tilde{v}_{j,z}$ is an analytic function of z. Since an exchange of critical points of first contact can only happen if at some point z the two critical points involved in the exchange are simultaneously contained in the critical curve $\tilde{L}_j = \tilde{L}_{j,z}$, and have therefore the same critical level. Consequently, the critical level $\tilde{c}_{j,z}$ is a continuous function of z, even if there is an exchange of critical points. Hence, an exchange can happen only if at points $z \in \mathbb{C}$, where different branches of the multivalued function $h \circ \pi^{-1}$ have identical values. We conclude that at least for all z in the open set $\tilde{D}$ defined by

$$\tilde{D} := \{ z \in \mathbb{C} \mid \operatorname{card}(h \circ \pi^{-1}(\{z\})) = 3 \} \tag{4.30}$$

the critical level $\tilde{c}_{j,z}$ is a harmonic function of z, and it is continuous for all $z \in \mathbb{C}$ in case of $j \neq \infty$, and for all $z \in \mathbb{C} \backslash \{0\}$ in case of $j = \infty$.

For $z \in D_j$ near infinity and near the origin we know that identity (4.25) holds true. We use the informations from Lemma 3.5 together with some conclusions from the proof of Theorem 2.8 for checking which continuations of the critical level $\tilde{c}_{j,z}$ as function of z are possible and compatible with the property of continuity in $\mathbb{C}$, or $\mathbb{C}\backslash\{0\}$, and harmonicity in $\tilde{D}$. The continuation is starting from a neighborhood of infinity. It then turns out that indeed identity (4.25) has two hold for all $z \in D_j$ and any choice of $j \in \{-1, 0, 1, \infty\}$. This immediately implies identity (4.26) for all $z \in D_j$.

Thus, for instance, in case of $j = 1$ it follows from Lemma 3.5 in combination with (3.72) that there exist in principle only two possibilities for the continuation of the critical level $\tilde{c}_{1,z}$ starting from a neighborhood of infinity. One is the function h_1 defined by (3.72), and the other one can be defined by

$$\tilde{h}_1(z) = \begin{cases} \hat{h}_0(z) & \text{for } z \in \overline{D_1^{(1)} \cup D_1^{(0)}}, \\ \hat{h}_1(z) & \text{for } z \in \mathbb{C}\backslash\overline{D_1^{(1)} \cup D_1^{(0)}} \end{cases} \tag{4.31}$$

with $\hat{h}_0$ and $\hat{h}_1$ being the same functions as that used in (3.72). Since $\tilde{h}_1$ is not bounded from below at the origin, there is a conflict with our earlier observation about the boundedness of $\tilde{c}_{1,z}$ from below for z in a neighborhood of the origin, and therefore the only possible continuation is h_1, which proves identity (4.25) for the case $j = 1$. Identity (4.26) is an immediate consequence.

The same conclusion follows for the case $j = -1$ by symmetry. For the cases $j = 0$ and ∞ identity (4.25) can be proved in an analogous way. The situation is even slightly simpler; instead of (3.72) one has now to use (3.78) and (3.83) from the proof of Theorem 2.8.

It remains to show that the critical curve $\tilde{L}_{j,z}$ contains only one critical point for all $z \in D_j$. It follows immediately from the definition of $\tilde{D}$ in (4.30) that this assertion is true for all $z \in \tilde{D}$. For $z \in D_j \backslash \tilde{D}$ we shall prove the same assertion indirectly. Let us assume that there exists $z_0 \in D_{j_1} \backslash \tilde{D}, j_1 \in \{-1, 0, 1, \infty\}$, such that the critical curve $\tilde{L}_{j_1,z_0}$ contains at least to different critical points v_{l_1,z_0} and $v_{l_2,z_0}, j_2 \in \{-1, 0, 1, \infty\}, j_2 \neq j_1$, and $z_0 \in D_{j_1} \cap D_{j_2}$. We define

$$d := h_{j_1} - h_{j_2}. \tag{4.32}$$

Since $d(z_0) = 0$, it follows from Lemma 3.2 that there exist an open, analytic Jordan arc $\tilde{\Gamma}$ with $z_0 \in \tilde{\Gamma}$, $d(z) = 0$ for all $z \in \tilde{\Gamma}$, and

$$\frac{\partial}{\partial n} d(z_0) \neq 0 \tag{4.33}$$

with $\partial/\partial n$ is the normal derivative with respect to the arc $\tilde{\Gamma}$. From (4.33) it follows that in any neighborhood $U \subset D_{j_1} \cap D_{j_2}$ of z_0 there exist two points $z_1, z_2 \in U$ such that

$$h_{j_1}(z_1) = \tilde{c}_{j_1,z_1} < h_{j_2}(z_1) = \tilde{c}_{j_2,z_1}$$

and

$$h_{j_1}(z_2) = \tilde{c}_{j_1,z_2} > h_{j_2}(z_2) = \tilde{c}_{j_2,z_2}. \tag{4.34}$$

From (4.34) we conclude that neither $v_{l_1,z}$ nor $v_{l_2,z}$ can remain a critical point of first contact on $\tilde{L}_{j_1,z}$ if the point z crosses $\tilde{\Gamma}$ at z_0. Hence, the critical level $\tilde{c}_{l_1,z}$ cannot be a harmonic function at $z = z_0$. This contradicts our assumption that $z_0 \in D_{j_1}$ and identity (4.26), which we have already proved. The contradiction proves the last part of the lemma.

We add the remark that the proof of the last part of the lemma has shown that a critical curve $\tilde{L}_{j,z}$ can contain more than one critical point only if $z \in \partial D_j$. Since we know from Theorem 2.8 that the function h_j is not harmonic at each point $z \in \partial D_j$, we also can conclude that for each $z \in \partial D_j$ the critical curve $\tilde{L}_{j,z}$ contains at least two different critical points. □

4.4. *Proof of Theorem 2.9*

Starting point of the proof is the representation of the three polynomials P_n, Q_n, R_n, and the remainder term E_n by integrals (1.9)–(1.12). The asymptotic relations (2.33)–(2.36) will be proved by applying the saddle point method to integrals (1.9)–(1.12), and using the integration paths $C_j, j = -1, 0, 1, \infty$, introduced in Lemma 4.6, and the transformations of the integrals introduced in Section 4.2.

All four asymptotic relations (2.33)–(2.36) are of similar structure. We first consider relation (2.33), i.e., the asymptotic relation for the Hermite–Padé polynomial P_n. The three other relations will be proved in a very analogous way. We set $j := -1$ and assume that $z \in D_{-1}$. From representation (1.9) and relation (4.8) together with the definition of the two integrals (4.5) and (4.7), we deduce that

$$P_n(z) = e^{3nz}\tilde{I}_n = e^{3nz} I_n\left(1 + O\left(\frac{1}{n}\right)\right) \quad \text{as } n \to \infty. \tag{4.35}$$

Landau's symbol in (4.35) holds uniformly for z varying in compact subsets of D_{-1}.

Let the functions g, G, and $\operatorname{Re} a_z(\cdot)$ be defined as in (4.9)–(4.11) and let $C_{-1} = C_{-1,z}$ be the integration path introduced in Lemma 4.6. Then the integral I_n introduced in (4.7) has the form

$$I_n = \int_{C_{-1}} g(v) G(v)^n \, dv, \tag{4.36}$$

which is exactly the form (4.1) that has been assumed in the Saddle Point Theorem.

Let $V \subseteq D_{-1}$ be an open neighborhood of the point z. From Lemma 4.6 it follows that assumption (ii) in the Saddle Point Theorem 4.1 is satisfied for all curves $C_{-1,z'}, z' \in V$, of Lemma 4.6. The critical points $v_{-1,z'}, z' \in V$, take over the role of the points $\zeta_0 = \zeta_{0,z'}, z' \in V$, in the formulation of the Saddle Point Theorem 4.1.

From (4.22) in Lemma 4.6 we know that condition (4.2) holds true. The analyticity requirements in (iii) are satisfied if we take $U := D_{-1}$. Thus, assumption (iii) of the Saddle Point Theorem 4.1 is also satisfied.

From Lemma 4.2 together with (4.9), (4.10), and (4.12) we deduce that also assumption (iv) of the Saddle Point Theorem 4.1 is satisfied.

After the check of the assumptions we know that Theorem 4.1 can be applied to integral (4.36), and from (4.3) we deduce that

$$I_n = \pm\sqrt{\frac{-2\pi G(v_{-1})}{nG''(v_{-1})}}g(v_{-1})G(v_{-1})^n\left(1+O\left(\frac{1}{n}\right)\right) \quad \text{as } n\to\infty \tag{4.37}$$

with a Landau symbol $O(1/n)$ that holds uniformly for z varying on compact subset of D_{-1}. From (4.12) we deduce that

$$G''(v) = G(v)[a_z'(v)^2 + a_z''(v)], \tag{4.38}$$

$$a_z''(v) = \frac{3v^4+1}{v^2(v^2-1)^2}. \tag{4.39}$$

Notice that z is kept fixed. For the critical point $v_{-1} = v_{-1,.z}$ we have

$$a_z'(v_{-1}) = 0. \tag{4.40}$$

From (4.38)–(4.39), and (4.40), it follows that

$$G''(v_{-1}) = G(v_{-1})\frac{3v_{-1}^4+1}{v_{-1}^2(v_{-1}^2-1)^2}. \tag{4.41}$$

Next we simplify the expressions in (4.37). From identities (4.41), (4.16), and (4.11) it follows that

$$\begin{aligned}\frac{-2\pi G(v_{-1})}{nG''(v_{-1})} &= \frac{-2\pi}{n}\frac{v_{-1}^2(v_{-1}^2-1)^2}{3v_{-1}^4+1} \\ &= \frac{-2\pi}{n}\frac{\psi_{-1}(z)^2(\psi_{-1}(z)^2-1)^2}{3\psi_{-1}(z)^4+1},\end{aligned} \tag{4.42}$$

$$g(v_{-1}) = \frac{-i\sqrt{2n}}{\sqrt{\pi}v_{-1}(v_{-1}^2-1)} = \frac{-i\sqrt{2n}}{\sqrt{\pi}\psi_{-1}(z)(\psi_{-1}(z)^2-1)}, \tag{4.43}$$

$$\sqrt{\frac{-2\pi G(v_{-1})}{nG''(v_{-1})}}g(v_{-1}) = \frac{\pm 2}{\sqrt{3\psi_{-1}(z)^4+1}}. \tag{4.44}$$

From (4.17) in Lemma 4.4 and the introduction of the function h_{-1}^* as the analytically completed form of h_{-1} in Definition 2.7, we conclude that

$$a_z(v_{-1,z}) = h_{-1}^*(z), \quad z\in D_{-1}. \tag{4.45}$$

Inserting identities (4.44) and (4.45) in the asymptotic relation (4.37) and using (4.35) yields

$$\begin{aligned} P_n(z) &= \frac{\pm 2e^{3nz}}{\sqrt{3\psi_{-1}(z)^4+1}} G(v_{-1,z})^n\left(1+O\left(\frac{1}{n}\right)\right) \\ &= \frac{\pm 2}{\sqrt{3\psi_{-1}(z)^4+1}} e^{n(h^*_{-1}(z)+3z)}\left(1+O\left(\frac{1}{n}\right)\right) \quad \text{as } n\to\infty. \end{aligned} \tag{4.46}$$

From (2.14) in Lemma 2.5 and from (3.1) in the proof of Lemma 2.5 we know that

$$h^*_{-1}(z) = -3z + \log z + O\left(\frac{1}{z}\right) \quad \text{as } z\to\infty, \tag{4.47}$$

$$\psi_{-1}(z) = -1 + O\left(\frac{1}{z}\right) \quad \text{as } z\to\infty. \tag{4.48}$$

Since P_n is monic, it follows from (4.47) and (4.48) that in (4.46) the plus sign has to be chosen, which proves the asymptotic relation (2.33).

The three other asymptotic relations (2.34)–(2.36) in the theorem can be proved in a nearly identical way as relation (2.33). Instead of relation (4.35) we have to use the relations

$$\begin{aligned} &Q_n(z) = I_{n,0}\left(1+O\left(\frac{1}{n}\right)\right), \quad R_n(z) = e^{-3nz}I_{n,1}\left(1+O\left(\frac{1}{n}\right)\right), \\ &E_n(z) = I_{n,\infty}\left(1+O\left(\frac{1}{n}\right)\right) \quad \text{for } n\to\infty \end{aligned} \tag{4.49}$$

with integrals $I_{n,j} = I_{n,j}(z), j = 0, 1, \infty$, defined as in (4.36), but now the integration paths $C_j = C_{j,z}$, $j = 0, 1, \infty$, are used instead of the integration path C_{-1}. The new paths have also been introduced in Lemma 4.6. In all other parts of the analysis one follows identical patterns. The necessary modifications should cause no difficulties. □

4.5. *Proof of Theorem 2.2*

In Remark 1 to Theorem 2.9 it has already been mentioned that the expressions $\sqrt{3\psi_j(z)^4+1}$ are analytic and different from zero for all $z\in D_j$ and $j = -1, 0, 1, \infty$. Hence, from the asymptotic estimates (2.33)–(2.36) in Theorem 2.9 we immediately conclude that

$$\frac{1}{n}\log|P_n(z)| = h_{-1}(z) + 3\,\mathrm{Re}(z) + O\left(\frac{1}{n}\right) \quad \text{as } n\to\infty,\ z\in D_{-1}, \tag{4.50}$$

$$\frac{1}{n}\log|Q_n(z)| = h_0(z) + \log 2 + O\left(\frac{1}{n}\right) \quad \text{as } n\to\infty,\ z\in D_0, \tag{4.51}$$

$$\frac{1}{n}\log|R_n(z)| = h_1(z) + 3\,\mathrm{Re}(z) + O\left(\frac{1}{n}\right) \quad \text{as } n\to\infty, z\in D_1, \tag{4.52}$$

$$\frac{1}{n}\log|E_n(z)| = h_\infty(z) + O\left(\frac{1}{n}\right) \quad \text{as } n\to\infty,\ z\in D_\infty. \tag{4.53}$$

As in Theorem 2.9 so also here the Landau symbols $O(\cdot)$ hold uniformly on compact subsets of the domains for which the corresponding asymptotic estimates are defined. In the derivation of (4.50)–(4.53) from (2.33)–(2.36) we have used $h_j = \mathrm{Re}\, h_j^*, j = -1, 0, 1, \infty$, which follows immediately from Definition 2.7.

From Theorem 2.8 we know that

$$D_j = \mathbb{C}\backslash\mathrm{supp}(v_j) \quad \text{for } j = -1, 0, 1, \infty. \tag{4.54}$$

Using representations (2.31) from Theorem 2.8, we can rewrite (4.50)–(4.52) as

$$\frac{1}{n}\log|P_n(z)| = \int \log|z-t|\,dv_{-1}(t) + O\left(\frac{1}{n}\right) \quad \text{as } n\to\infty,\ z\in D_{-1}, \tag{4.55}$$

$$\frac{1}{n}\log|Q_n(z)| = \log 2 + \int \log|z-t|\,dv_0(t) + O\left(\frac{1}{n}\right) \quad \text{as } n\to\infty,\ z\in D_0, \tag{4.56}$$

$$\frac{1}{n}\log|R_n(z)| = \int \log|z-t|\,dv_1(t) + O\left(\frac{1}{n}\right) \quad \text{as } n\to\infty,\ z\in D_1, \tag{4.57}$$

which hold with the same Landau symbols $O(\cdot)$ as those used in (4.50)–(4.53). The first three limits in (2.3) then follow immediately from (4.55)–(4.57) together with (4.54).

From assertion (iii) of Theorem 2.8 we know that $h_\infty = h(v_\infty;\cdot)$. Hence, we can deduce from (4.53) and (4.54) that

$$\frac{1}{n}\log|E_n(z)| = h(v_\infty; z) + O\left(\frac{1}{n}\right) \quad \text{as } n\to\infty,\ z\in\mathbb{C}\backslash(\mathrm{supp}(v_\infty)\cup\{0\}), \tag{4.58}$$

where again the Landau symbol $O(\cdot)$ holds uniformly on compact subsets of the domain $\mathbb{C}\backslash(\mathrm{supp}(v_\infty)\cup\{0\})$. From (4.58) the fourth limit in (2.3) follows immediately. □

4.6. *Proof of Theorem 2.1*

We start by considering the first limit in (2.2). In (1.7) the polynomials P_n have been assumed to be monic and of exact degree n. The admissibility of this assumption has been justified by the perfectness of the exponential function, which has been shown in [15]. At this late stage of our investigation we can justify this assumption for n sufficiently large also by the asymptotic estimate (2.33) in Theorem 2.9.

It has already been mentioned in the proof of Theorem 2.2 that the expression $\sqrt{3\psi_{-1}(z)^4+1}$ is different from zero for all $z\in D_{-1}$. It therefore follows from the asymptotic relation (2.33) in Theorem 2.9 that all zeros of the polynomials P_n cluster on the arc $\Gamma_{-1}=\partial D_{-1}$ if $n\to\infty$. Let $Z(P_n)$ denote the (multi-)set of all zeros of P_n. Then we have

$$\bigcap_{n=1}^{\infty}\overline{\bigcup_{m\geqslant n} Z(P_m)}\subseteq\Gamma_{-1}. \tag{4.59}$$

Because of (4.59) the sequence $\{\frac{1}{n}\nu_{P_n}\}_{n=1}^{\infty}$ of probability measures is compact with respect to the weak convergence of measures (Helly's Selection Theorem, cf. [19, Theorem 1.3]). Let ν_{-1}^* denote a cluster point of the sequence $\{\frac{1}{n}\nu_{P_n}\}_{n=1}^{\infty}$, and assume that $N\subseteq\mathbb{N}$ is an infinite subsequence with

$$\frac{1}{n}\nu_{P_n}\overset{*}{\to}\nu_{-1}^*\quad\text{as } n\to\infty,\ n\in N. \tag{4.60}$$

From (4.59) it follows that $\operatorname{supp}(\nu_{-1}^*)\subseteq\Gamma_{-1}$. We have

$$\frac{1}{n}\log|P_n(z)|=\frac{1}{n}\int\log|z-t|\,d\nu_{P_n}(t). \tag{4.61}$$

From (4.61), limit (4.60), and the first limit in (2.3) of Theorem 2.2 it follows that

$$\int\log|z-t|\,d\nu_{-1}^*(t)=\int\log|z-t|\,d\nu_{-1}(t)\quad\text{for } z\in D_{-1}. \tag{4.62}$$

Since $\operatorname{supp}(\nu_{-1}^*)\subseteq\partial D_{-1}$, it is a standard conclusion from potential theory (Carleson's Unicity Theorem. cf. [19, Theorem 4.13]) that identity (4.62) implies that

$$\nu_{-1}^*=\nu_{-1}. \tag{4.63}$$

The measure ν_{-1} is independent of the subsequence N, and consequently limit (4.60) holds for the full sequence $\{\frac{1}{n}\nu_{P_n}\}_{n=1}^{\infty}$, which together with (4.63) proves the first limit in (2.2).

The third limit in (2.2) follows in exactly in the same way as the first one, only the role of the measures ν_1 and ν_{-1} has to be interchanged. We note that the polynomials R_n are the image of the polynomials P_n under reflections on the imaginary axis, which implies that the polynomials R_n are all monic and of exact degree n.

The polynomials Q_n are not monic. If in representation (1.10) we replace e^{3nzv} by its power series and apply the residue theorem to the integral in (1.10), then for the leading coefficient in Q_n we get $(-2)^{n+1}$, i.e., we have

$$Q_n(z)=(-2)^{n+1}z^n+\cdots \tag{4.64}$$

As done before (4.59) so we can also here conclude that all zeros of the polynomials Q_n cluster on the set K_0 if $n\to\infty$, and we have a relation analogous to (4.59) with Γ_{-1} replaced by K_0. The sequence $\{\frac{1}{n}\nu_{Q_n}\}_{n=1}^{\infty}$ of positive measures is compact with respect to weak convergence. Let ν_0^* be a cluster point of this sequence, then there exists an infinite subsequence $N \subseteq \mathbb{N}$ such that

$$\frac{1}{n} v_{Q_n} \xrightarrow{*} v_0^* \quad \text{as } n \to \infty, \ n \in N. \tag{4.65}$$

We have $\text{supp}(v_0^*) \subseteq K_0$. From (4.64), (4.65), and the second limit in (2.3) of Theorem 2.2 it then follows that

$$\int \log |z - t| \, dv_0^*(t) = \int \log |z - t| \, dv_0(t) \quad \text{for } z \in D_0. \tag{4.66}$$

With the same arguments as used after (4.62) it then follows from (4.66) that

$$v_0^* = v_0, \tag{4.67}$$

from which the second limit in (2.2) follows.

It remains to prove that also the fourth limit in (2.2) holds true. Here, the procedure is more involved since the support of the measure v_∞ is unbounded, its mass is infinite, and the remainder term E_n has infinitely many zeros. As in representations (2.4) and (2.32) we confine our investigations to disks $\{|z| \leqslant R\}$.

Let $R > 0$ be chosen arbitrary, but sufficiently large so that $\Gamma_{-1} \cup K_0 \cup \Gamma_1 \subseteq \{|z| < R\}$. By $Z(E_n, R)$ we denote the (multi-) set of zeros of the remainder term E_n in $\{|z| < R\}$, by $v_{E_n,R}$ the corresponding zero counting measure, and by

$$b_n(z) := \prod_{x \in Z(E_n,R)} \frac{R(z - x)}{R^2 - \bar{x}z} \tag{4.68}$$

the Blaschke product in $\{|z| < R\}$ that has the same zeros as E_n in $\{|z| < R\}$.

From Remark 2 to Definition 2.7 we know that the three domains D_{-1}, D_0, and D_1 all cover the circle $\{|z| = R\}$. Therefore, it follows from the first three limits in (2.3) of Theorem 2.2, from the definition of E_n in (1.6), from (2.18), and from (3.84) that for any $\epsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that

$$\begin{aligned} \frac{1}{n} \log |E_n(z)| &\leqslant (1 + \epsilon) \max(h_{-1}(z), h_0(z), h_1(z)) \\ &= (1 + \epsilon) h_\infty(z) \quad \text{for } |z| = R \text{ and } n \geqslant n_0. \end{aligned} \tag{4.69}$$

From (4.68) and (4.69) we see that the functions $\sqrt[n]{E_n(z)/b_n(z)}$ form a normal family in $\{|z| < R\}$.

From the asymptotic estimate (2.36) in Theorem 2.9 it follows that all zeros of the remainder term E_n in $\{|z| \leqslant R\}$ cluster on the set $(K_\infty \cap \{|z| < R\}) \cup \{0\}$ if $n \to \infty$. Let $v_{\infty,R}^*$ be a cluster point of the sequence $\{\frac{1}{n} v_{E_n,R}\}_{n=1}^\infty$ of positive measures. We then have $\text{supp}(v_{\infty,R}^*) \subseteq \{0\} \cup (K_\infty \cap \{|z| \leqslant R\})$. At this stage we do not know whether $v_{\infty,R}^*$ is of finite mass, but we can conclude, as in Helly's Selection Theorem, that there exists an infinite subsequence $N \subseteq \mathbb{N}$ such that

$$\frac{1}{n} v_{E_n,R} \xrightarrow{*} v_{\infty,R}^* \quad \text{as } n \to \infty, \ n \in N. \tag{4.70}$$

From Montel's theorem we know that we can choose the subsequence $N \subseteq \mathbb{N}$ in such a way that besides of (4.70) also the limit

$$\limsup_{n\to\infty, n\in N} \left| \frac{1}{n} \log |E_n(z)| - \frac{1}{n} \int \log \left| \frac{R(z-t)}{R^2 - \bar{t}z} \right| d\nu_{E_n,R}(t) \right] =: g(z) \tag{4.71}$$

exists locally uniformly in $\{|z| < R\}$. It is immediate that the function g is harmonic in $\{|z| < R\}$. From the upper bound (4.69) and the behavior of b_n in a neighborhood of $\{|z| = R\}$ we conclude that

$$\limsup_{z'\to z, |z'|<R} g(z') \leqslant h_\infty(z) \quad \text{for } |z| = R, \tag{4.72}$$

which together with the definition of the function h_R in (2.20) implies that

$$g(z) \leqslant h_R(z) \quad \text{for } |z| < R. \tag{4.73}$$

From (4.70), (4.71), the fourth limit in (2.3) of Theorem 2.2, and (4.73) we conclude that

$$\int \log \left| \frac{R(z-t)}{R^2 - \bar{t}z} \right| d\nu^*_{\infty,R}(t) = h_\infty(z) - g(z)$$
$$\geqslant h_\infty(z) - h_R(z) \quad \text{for } z \in (\{|z| < R\} \cap D_\infty) \setminus \{0\}. \tag{4.74}$$

From representation (2.32) in Theorem 2.8 we know that

$$(h_\infty - h_R)(z) = \int_{|t|<R} \log \left| \frac{R(z-t)}{R^2 - \bar{t}z} \right| d\nu_\infty(t) + 3 \log \frac{|z|}{R} \quad \text{for } |z| < R. \tag{4.75}$$

Putting (4.74) and (4.75) together yields

$$(h_R - g)(z) = (h_\infty(z) - g(z)) - (h_\infty(z) - h_R(z))$$
$$= \int \log \left| \frac{R(z-t)}{R^2 - \bar{t}z} \right| d(\nu^*_{\infty,R} - \nu_\infty|_{\{|z|<R\}} - 3\delta_0)(t) \tag{4.76}$$

for $z \in (\{|z| < R\} \cap D_\infty) \setminus \{0\}$. Since the left-hand side of (4.76) is harmonic in $\{|z| < R\}$, it follows from the right-hand side of (4.76) that

$$\nu^*_{\infty,R}|_{\{|z|<R\}} = \nu_\infty|_{\{|z|<R\}} + 3\delta_0. \tag{4.77}$$

From the arbitrariness of the constant R and from the fact that the right-hand side of (4.77) is independent of the subsequence $N \subseteq \mathbb{N}$, we conclude that limit (4.70) holds in the form

$$\frac{1}{n} \nu_{E_n,R} \xrightarrow{*} \nu_\infty|_{\{|z|\leqslant R\}} + 3\delta_0 \quad \text{as } n \to \infty, \tag{4.78}$$

which proves the fourth limit in (2.2). This completes the proof of Theorem 2.1. □

4.7. Appendix to the Proof of Theorem 2.8

In Section 3, we have proved Theorem 2.8 only with the exception of the remark that the measures $\nu_j, j = -1, 0, 1, \infty$, defined in (2.28) and (2.29) are the same as those appearing in the Theorems 2.1 and 2.2. This remark has been made in assertion

(i) of Theorem 2.8. Now, after the proofs of Theorems 2.1 and 2.2 we know that this remark is correct. We will show in this appendix that no circular argument has been applied.

Indeed, all results proved in Section 3 and also all results proved in the present section up to this point were based on the analytic definition of the measures $\nu_j, j = -1, 0, 1, \infty$, by (2.28) and (2.29). This definition uses only the Riemann surface $\mathcal{R}$ and the function h introduced in Definitions 2.3 and 2.4, respectively. Thus, the measures ν_j have been defined by geometrical considerations only. The asymptotic behavior of the Hermite–Padé polynomials P_n, Q_n, R_n, and the remainder term E_n has played no role in this definition. □

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